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Steiner intervals, geodesic intervals, and betweenness[☆]

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ABSTRACT

The concept of the k-Steiner interval is a natural generalization of the geodesic (binary) interval. It is defined as a mapping $S : V \times \cdots \times V \longrightarrow 2^V$ such that $S(u_1, \ldots, u_k)$ consists of all vertices in G that lie on some Steiner tree with respect to a multiset $W = \{u_1, \ldots, u_k\}$ of vertices from G. In this paper we obtain, for each k, a characterization of the class of graphs in which every k-Steiner interval S has the so-called union property, which says that $S(u_1, \ldots, u_k)$ coincides with the union of geodesic intervals $I(u_i, u_j)$ between all pairs from W. It turns out that, as soon as $k > 3$, this class coincides with the class of graphs in which the k-Steiner interval enjoys the monotone axiom (m), respectively (b2) axiom, the conditions from betweenness theory. Notably, S satisfies (m), if $x_1, \ldots, x_k \in S(u_1, \ldots, u_k)$ implies $S(x_1, \ldots, x_k) \subseteq S(u_1, \ldots, u_k)$, and S satisfies (b2) if $x \in S(u_1, u_2, \ldots, u_k)$ implies $S(x, u_2, \ldots, u_k) \subseteq S(u_1, \ldots, u_k)$. In the case $k = 3$, these three classes are different, and we give structural characterizations of graphs for which their Steiner interval S satisfies the union property as well as the monotone axiom (m). We also prove several partial observations on the class of graphs in which the 3-Steiner interval satisfies (b2), which lead to the conjecture that these are precisely the graphs in which every block is a geodetic graph with diameter at most two.

1. Introduction and preliminaries

For vertices u, v of a graph G, the interval $I(u, v)$ in graphs is usually defined as the set of vertices lying on a *geodesic* (shortest path) between u and v. But besides geodesics there are some other notions that can be used for defining an interval, such as induced and detour paths. In this paper we will focus on yet another concept, the Steiner interval, and consider its relation with geodesic intervals.

The Steiner tree problem is a well-known problem with several variations and applications. It can concern points in Euclidean (or other metric) spaces, and vertices of weighted or non-weighted graphs [11], and has drawn much attention

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due to the development of approximation algorithms, see [1,16] and the references therein. In a non-weighted connected graph G, a *Steiner tree* of a (multi)set $W \subseteq V(G)$, is a minimum order tree in G that contains all vertices of W. The number of edges in a Steiner tree T of W is called *the Steiner distance of* W, denoted $d(W)$, while *the size* of T describes the number of vertices in T (i.e. $d(W) + 1$). The *k-Steiner interval* is a mapping $S : V \times \cdots \times V \longrightarrow V$ such that $S(u_1, u_2, \ldots, u_k)$ consists of all vertices in G that lie on some Steiner tree with respect to $\{u_1, \ldots, u_k\}$, where $u_1, \ldots, u_k$ are, not necessarily distinct, vertices of G (in this way S is an extension of I, as $S(u, v, \ldots, v) = I(u, v)$). (Note that, as above, we will simplify the notation for $S(\{u_1, \ldots, u_k\})$ to $S(u_1, \ldots, u_k)$, where S denotes the k-Steiner interval, and $u_1, \ldots, u_k$ are not necessarily distinct vertices of a graph.) Steiner intervals on ordinary vertex subsets have been studied in several papers [6,8–11,13,16,20–22].

One of the main issues regarding Steiner intervals is related to connections between different variations of the geodetic number, see the survey paper [5]. Chartrand and Zhang proposed a natural concept of the Steiner number of a graph [6], and among several nice results "proved" an erroneous statement [6] regarding the connection between Steiner intervals of a set of vertices, and the union of geodesic intervals between pairs of the vertices from the set. This error was observed and corrected by Pelayo [22], and the development intrigued Hernando et al. [10] to raise the following problem: for which graphs the Steiner interval of any set of vertices, whose union of geodesic intervals between pairs of vertices in the set is the whole vertex set, also yields all vertices? Certainly this property holds for graphs in which $S(W) \subseteq \cup_{u,v\in W} I(u, v)$ for all $W \subseteq V(G)$, which was shown to be true in distance hereditary graphs [10,21] and in a more general family of 3-Steiner distance hereditary graphs [8]. A characterization of these graphs (in which $S(W) \subseteq \cup_{u,v\in W} I(u, v)$ for all $W \subseteq V(G)$) remains open, and seems to be quite difficult. In this paper we consider the following stronger condition: given a fixed k, for any multiset W of vertices with $|W| = k$,

$$S(W) = \bigcup_{u,v\in W} I(u, v).$$

We call this *the union property* of the k-Steiner interval. When $k = 2$ the union property trivially holds in all graphs. We prove in Section 2 that for any k greater than 3, the union property holds precisely in block graphs. The case $k = 3$ turns out to be the most difficult and interesting, see Section 3. (See also [9,13,20] for other studies on Steiner intervals.)

The second focus of this paper is on the concept of betweenness in graphs as introduced by Mulder in [18] (he implicitly considered this notion already in his book [17]). Starting points of his study were two very strong properties that the geodesic interval I enjoys. Namely, (i) if x is between u and v (i.e. $x \in I(u, v)$) and $x \neq u$, then u is not between x and v, and, (ii) if x is between u and v, and y is between u and x, then y is between u and v. Properties (i) and (ii) are usually denoted by (b1) and (b2), respectively, and together they form *betweenness axioms* as defined in [18]. The interpretation of the betweenness properties in this sense was first studied by Mulder and Morgana [15], where it was also proved that the induced path interval J is a betweenness if and only if G is a house, hole, domino-free graph. A related property (not always satisfied by I) is that if x and y are between u and v, and z is between x and y, then z is between u and v. This property is known as the *monotone axiom*, which is again introduced formally in [18]. (In [26], van de Vel uses the term monotone law for what we call the (b2) axiom.) The graphs in which the monotone axiom is always satisfied are known as interval monotone graphs which were also introduced in [17], see also [2,14]. Clearly the monotone axiom always implies (b2), but the converse need not hold and the characterization of interval monotone graphs is still an open problem. Betweenness in discrete structures other than graphs has been studied much earlier, for example see [24].

As 2-Steiner intervals are precisely the geodesic intervals I, k-Steiner intervals form a generalization of the geodesic interval, hence it is natural to look at the analogous concept of betweenness for k-Steiner intervals. The betweenness axioms and the monotone axiom (m) can be generalized in a natural way from binary to k-ary functions (in particular from geodesic intervals to k-Steiner intervals) as follows: for any $u_1, u_2, \ldots, u_k, x, x_1, x_2, \ldots, x_k \in V(G)$ which are not necessarily distinct,

(b1) $x \in S(u_1, u_2, \ldots, u_k), x \neq u_1 \Rightarrow u_1 \notin S(x, u_2, \ldots, u_k)$,
(b2) $x \in S(u_1, u_2, \ldots, u_k) \Rightarrow S(x, u_2, \ldots, u_k) \subseteq S(u_1, u_2, \ldots, u_k)$,
(m) $x_1, x_2, \ldots, x_k \in S(u_1, u_2, \ldots, u_k) \Rightarrow S(x_1, x_2, \ldots, x_k) \subseteq S(u_1, u_2, \ldots, u_k)$.

Somewhat surprisingly for the k-Steiner interval, where $k > 2$, the betweenness axioms are not satisfied in all graphs. As we show in Section 3, in the case $k = 3$ the class of graphs in which the 3-Steiner interval has the union property (which are the graphs in which each block is a clique or a 5-cycle) is properly contained in the class of graphs in which the 3-Steiner interval satisfies the monotone axiom (m), which is in turn properly contained in the class of graphs in which the 3-Steiner interval satisfies (b2). Example of graphs, for which the monotone axiom is satisfied for the 3-Steiner interval S but not the union property are the graphs M_k, $k \geq 3$, see Fig. 8. An example of a graph for which the 3-Steiner interval S satisfies (b2), but not (m) is the famous Petersen graph, see Fig. 10. One can easily verify that (m) is not satisfied by the Petersen graph. By using the labeling of vertices from Fig. 10, note that $S(b, d, f)$ consists of all vertices in the graph except for w, yet $w \in S(x, y, z)$. Hence $S(x, y, z) \nsubseteq S(b, d, f)$, and (m) is not satisfied for S. On the other hand, for any k greater than 3 the classes of graphs in which the k-Steiner interval satisfies the union property, the monotone axiom, and the (b2) axiom are all the same, which is the main theorem in Section 2.

We conclude this section with the following lemma that considerably reduces the class of graphs in which the 3-Steiner interval satisfies the union property. This property is even more restrictive for the k-Steiner interval where $k > 3$. Recall that a subgraph H of a graph G is an *isometric subgraph* of G if for any pair of vertices $u, v \in V(H)$, there exists a geodesic