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Finite Euclidean graphs and Ramanujan graphs☆

Eiichi Bannai, Osamu Shimabukuro [1], Hajime Tanaka *

Graduate School of Mathematics, Kyushu University, Hakozaki 6-10-1, Higashi-ku, Fukuoka 812-8581, Japan

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ABSTRACT

We consider finite analogues of Euclidean graphs in a more general setting than that considered in [A. Medrano, P. Myers, H.M. Stark, A. Terras, Finite analogues of Euclidean space, J. Comput. Appl. Math. 68 (1996) 221–238] and we obtain many new examples of Ramanujan graphs. In order to prove these results, we use the previous work of [W.M. Kwok, Character tables of association schemes of affine type, European J. Combin. 13 (1992) 167–185] calculating the character tables of certain association schemes of affine type. A key observation is that we can obtain better estimates for the ordinary Kloosterman sum $K(a, b; q)$. In particular, we always achieve $|K(a, b; q)| < 2\sqrt{q}$, and $|K(a, b; q)| \leq 2\sqrt{q-2}$ in many (but not all) of the cases, instead of the well known $|K(a, b; q)| \leq 2\sqrt{q}$. Also, we use the ideas of controlling association schemes, and the Ennola type dualities, in our previous work on the character tables of commutative association schemes. The method in this paper will be used to construct many more new examples of families of Ramanujan graphs in the subsequent paper.

0. Introduction

The purpose of this paper is to continue the study on finite analogues of Euclidean graphs which was started in Medrano et al. [15].

In [15], the authors considered the following finite Euclidean graphs. Let $V = V_n(q) = \mathbb{F}_q^n$ be the n-dimensional vector space over the finite field $\mathbb{F}_q$ where $q = p^r$ with p being a prime number. (In [15], p was assumed to be an odd prime.) For $x, y \in V$, the Euclidean distance $d(x, y) \in \mathbb{F}_q$ is defined by

$$d(x, y) = (x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2.$$

The Euclidean graph $E_q(n, a)$ was defined as the graph with vertex set V and edge set

$$E = \{(x, y) \in V \times V \mid x \neq y,\ d(x, y) = a\}.$$

Then they considered the spectra of the graph $E_q(n, a)$ and discussed when these graphs are Ramanujan graphs. As is well known, a regular (undirected) graph of valency k is called a Ramanujan graph if any eigenvalue θ of the graph with $|\theta| \neq k$ satisfies

$$|\theta| \leq 2\sqrt{k-1}. \quad (1)$$

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* Corresponding address: Graduate School of Information Sciences, Tohoku University, Sendai 980-8579, Japan.
E-mail addresses: bannai@math.kyushu-u.ac.jp (E. Bannai), shimabukuro@fukushima-nct.ac.jp (O. Shimabukuro), htanaka@math.is.tohoku.ac.jp (H. Tanaka).

[1] Current address: Fukushima National College of Technology, Iwaki, Fukushima 970-8034, Japan.

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However, we remark that it is more natural to define finite analogues of Euclidean graphs for each non-degenerate quadratic form on V, instead of considering only the above distance $d(x, y) = (x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2$. That is, let Q be a non-degenerate quadratic form on V. Then the graph $E_q(n, Q, a)$ is defined as the graph with vertex set V and edge set

$$E = \{(x, y) \in V \times V \mid x \neq y,\ Q(x - y) = a\}. \quad (2)$$

The advantage of using this new definition is twofold. First, note that if $n = 2m$ is even, there are two inequivalent non-degenerate quadratic forms $Q = Q^{\pm}_{2m}$ on $V = \mathbb{F}_q^n$ (see Section 1). If $q \equiv 3 \pmod 4$, the quadratic form $d(x, 0) = x_1^2 + x_2^2 + \cdots + x_n^2$ is equivalent to $Q^{\mp}_{2m}$ depending on whether m is odd or even, while it is always equivalent to Q^{+}_{2m} if $q \equiv 1 \pmod 4$. Therefore, we obtain more examples of interesting graphs in a unified manner. Second, this allows us to consider finite analogues of Euclidean graphs when q is even. (In this case, $d(\cdot, 0)$ is degenerate. It is remarked in Medrano et al. [15] and Terras [17] that finite analogues of Euclidean graphs for $\mathbb{F}_q$, with q even, had not been studied.) Moreover, we will be able to see that the previous work of Kwok [10] is readily applicable when this new viewpoint is introduced. In fact, using the character tables of certain association schemes of affine type obtained in Kwok [10], we can obtain many new examples of Ramanujan graphs among the graphs $E_q(n, Q, a)$. (The reader is referred to Bannai–Ito [2] and Bannai [1] for the basic concept of commutative association schemes and their character tables.) We also remark that this phenomenon is closely connected with the previous work of Ennola type dualities in some association schemes given in Bannai–Kwok–Song [3], as we will discuss more in the subsequent papers (cf. [6]).

The content of this paper is as follows. In Section 1, we review basic materials on the character tables of association schemes which give the framework for the study of finite Euclidean graphs $E_q(n, Q, a)$. In particular, we will review the work of Kwok [10]. In Section 2, we consider certain Kloosterman sums (which are essentially Gaussian periods) and the well known inequality $|K(a, b; q)| \leq 2\sqrt{q}$ due to A. Weil [18]. Then we will discuss when the equality $|K(a, b; q)| = 2\sqrt{q}$ is attained. (We will show that this is never attained for the Kloosterman sums $K(a, b; q)$ we are considering.) Then, in Section 3, this result will be applied to discuss which of the finite Euclidean graphs $E_q(n, Q, a)$ are Ramanujan graphs. Our main results in this paper are Theorems 3.1–3.4 in Section 3. In addition, in Section 4, we will give some results of calculations by computer for some small parameters m and q, implementing the earlier work of Medrano et al. [15].

In order to keep this paper concise, we confined our discussions to the finite Euclidean graphs $E_q(n, Q, a)$. However, it is possible to obtain similar kinds of results for other many association schemes considered in the papers Bannai–Shen–Song [4], Bannai–Shen–Song–Wei [5], etc. (i.e., finite analogues of non-Euclidean graphs in the sense of [17, Chapter 19]). That study will be treated in the subsequent papers (cf. [6]).

1. Orthogonal groups, association schemes of certain affine types and their character tables

Let $\mathfrak{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be a commutative association scheme of class d (see Bannai [1] or Bannai–Ito [2] for instance). Let A_i be the adjacency matrix with respect to the relation R_i. Then $A_0, A_1, \ldots, A_d$ generate a semisimple algebra $\mathcal{A}$ over the complex number field, called the Bose–Mesner algebra of $\mathfrak{X}$. Let $E_0 = \frac{1}{|X|}J, E_1, \ldots, E_d$ be a unique set of primitive idempotents of $\mathcal{A}$, where J is the matrix whose entries are all 1, and write

$$A_i = \sum_{j=0}^{d} p_i(j) E_j,$$

for $0 \leq i \leq d$ (in particular, $k_i = p_i(0)$ is the valency of the regular graph (X, R_i)). The $(d + 1) \times (d + 1)$ matrix P whose (j, i)-entry is $p_i(j)$, is called the character table of the commutative association scheme $\mathfrak{X}$.

Many examples of association schemes are obtained as follows. Let G be a finite group acting transitively on a finite set X, and let $\mathcal{O}_0 = \{(x, x) | x \in X\}, \mathcal{O}_1, \ldots, \mathcal{O}_d$ be the orbits of G acting on the set $X \times X$. Define the relation R_i on X by

$$(x, y) \in R_i \Leftrightarrow (x, y) \in \mathcal{O}_i,$$

for $0 \leq i \leq d$, then it is easily seen that the pair $(X, \{R_i\}_{0 \leq i \leq d})$ is an association scheme.

Now, let Q be a non-degenerate quadratic form on the vector space $V = V_n(q) = \mathbb{F}_q^n$. Then the group of all linear transformations on V that fix Q, is called the orthogonal group associated with the quadratic form Q, and is denoted by $O(V, Q)$. More precisely,

$$O(V, Q) = \{\sigma \in GL(V) \mid Q(\sigma(x)) = Q(x) \text{ for all } x \in V\}.$$

The non-degenerate quadratic forms over $\mathbb{F}_q$ are classified as follows:

(i) Suppose $n = 2m$ is even. If q is odd, then there are two inequivalent non-degenerate quadratic forms Q^+ and Q^-:

$$Q^+(x) = 2x_1x_2 + \cdots + 2x_{2m-1}x_{2m},$$
$$Q^-(x) = 2x_1x_2 + \cdots + 2x_{2m-3}x_{2m-2} + x_{2m-1}^2 - \alpha x_{2m}^2,$$