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Blockers and transversals

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ABSTRACT

Given an undirected graph $G = (V, E)$ with matching number $\nu(G)$, we define d-blockers as subsets of edges B such that $\nu((V, E \setminus B)) \leq \nu(G) - d$. We define d-transversals T as subsets of edges such that every maximum matching M has $|M \cap T| \geq d$. We explore connections between d-blockers and d-transversals. Special classes of graphs are examined which include complete graphs, regular bipartite graphs, chains and cycles and we construct minimum d-transversals and d-blockers in these special graphs. We also study the complexity status of finding minimum transversals and blockers in arbitrary graphs.

1. Introduction

In this paper we introduce the following two concepts: in an undirected graph $G = (V, E)$ a set of edges T such that each maximum matching in G contains at least a given number d of edges of T is a d-transversal; a d-blocker is a set of edges B such that the matching number (the cardinality of a maximum matching) of $(V, E \setminus B)$ is at most the matching number of G minus d. We will consider the problem of finding a minimum d-transversal T and a minimum d-blocker B in G.

The problem of the d-blocker is closely related to some edge deletion and edge modification problems which have been studied in [4,13,14]. Similar problems have also been analyzed for vertices (see [5,12,15]).

In [10,11], the authors consider the problem of existence of a maximum matching whose removal leads to a graph with given upper (resp. lower) bound for the cardinality of its maximum matching. Here we will not impose any structure on the edge set representing the d-blocker.

In [2] a minimal blocker for a bipartite graph G is defined as a minimal set of edges the removal of which leaves no perfect matching in G and explicit characterizations of minimal blockers of bipartite graphs are given. An efficient algorithm enumerating the minimal blockers is given.

A concept close to d-transversal can be found in [3] where authors consider the notion of multiple transversal, another generalization of a transversal in the hypergraph of perfect matchings. Here a multiple transversal must intersect each perfect matching M_i with at least b_i edges.

A different concept of d-transversals has been studied in [6]. Given a set of integers $\{p_0, p_1, \ldots, p_s\}$ and a bipartite graph G, one has to find a minimum set of edges R such that for each p_i, $i = 0, 1, \ldots, s$, there exists a maximum matching M_i with $|M_i \cap R| = p_i$. Results have been given for special classes of bipartite graphs.

For some applications of the concept of colored blockers and transversals we refer the reader to [8].

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Our paper is organized as follows. In Section 2, we give some definitions and show some basic properties concerning transversals and blockers. We also study the connections between both notions. Section 3 deals with complexity results. We show that given two integers d and k, deciding whether there exists a d-transversal or a d-blocker of size k is $\mathcal{NP}$-complete in bipartite graphs. Some special classes of graphs are analyzed in Section 4. These include complete graphs, regular bipartite graphs, chains and cycles.

2. Definitions and basic properties

All graph theoretical terms not defined here can be found in [1]. Throughout this paper we are concerned with undirected simple loopless graphs $G = (V, E)$. The degree of a vertex v is denoted $d(v)$ and $\Delta(G)$ stands for the maximum degree of a vertex in G. G will be assumed connected. A **cut-edge** $e = uv$ is an edge such that its removal disconnects G. A **matching** M is a set of pairwise non-adjacent edges. A matching M is called **maximum** if its cardinality $|M|$ is maximum. The largest cardinality of a matching in G, its **matching number**, will be denoted by $\nu(G)$. More specifically we will be interested in subsets of edges which will intersect maximum matchings in G or whose removal will reduce by a given number the matching number.

We shall say that a subset $T \subseteq E$ is a ***d*-transversal** of G if for every maximum matching $M \in G$ we have $|M \cap T| \geq d$. Thus a d-transversal is a subset of edges which intersect each maximum matching in at least d edges.

A subset $B \subseteq E$ will be called a ***d*-blocker** of G if $\nu(G') \leq \nu(G) - d$ where G' is the partial graph $G' = (V, E \setminus B)$. So B is a subset of edges such that its removal reduces by at least d the cardinality of a maximum matching.

In case where $d = 1$, a d-transversal or a d-blocker is called a **transversal** or a **blocker**, respectively. We remark that in this case our definition of a transversal coincides with the definition of a transversal in the hypergraph of maximum matchings of G.

We denote by $\boldsymbol{\beta_d(G)}$ the minimum cardinality of a d-blocker in G and by $\boldsymbol{\tau_d(G)}$ the minimum cardinality of a d-transversal in G ($\beta(G)$ and $\tau(G)$ in case of a blocker or a transversal).

Let v be a vertex in graph G. The **bundle** of v, denoted by $\omega(v)$, is the set of edges which are incident to v. So $|\omega(v)| = d(v)$ is the degree of v. As we will see, bundles play an important role in finding d-transversals and d-blockers.

Let $\boldsymbol{P_0(G)} = \{vw \in E | \ \forall$ maximum matching $M, vw \notin M\}$ and $\boldsymbol{P_1(G)} = \{vw \in E | \ \forall$ maximum matching $M, vw \in M\}$. Let M be a matching. A vertex $v \in V$ is called **saturated by** M if there exists an edge $vw \in M$. A vertex $v \in V$ is called **strongly saturated** if for all maximum matchings M, v is saturated by M. We denote by $\boldsymbol{S(G)}$ the set of strongly saturated vertices of a graph G.

Notice that the sets $P_0(G)$, $P_1(G)$ and $S(G)$ can be determined in polynomial time. In fact, if we want to test whether an edge vw belongs to $P_0(G)$, we delete all edges having exactly one endpoint in $\{v, w\}$ and we determine a maximum matching M in the remaining graph. Then vw belongs to $P_0(G)$ if and only if $|M| = \nu(G) - 1$. In order to check whether an edge vw is in $P_1(G)$, we simply delete this edge and find a maximum matching M in the remaining graph. Then vw belongs to $P_1(G)$ if and only if $|M| = \nu(G) - 1$. By performing these tests for all edges in G, we determine the sets $P_0(G)$ and $P_1(G)$. Since a maximum matching in a graph can be found in polynomial time (see [7]), $P_0(G)$ and $P_1(G)$ can be determined in polynomial time. Concerning $S(G)$, first notice that all vertices which are incident to an edge of $P_1(G)$ necessarily belong to $S(G)$. For each other vertex v, to check whether it is strongly saturated, we simply delete it in G and find a maximum matching M in the remaining graph. Then v must belong to $S(G)$ if and only if $|M| = \nu(G) - 1$.

Remark 2.1. If G is a graph such that $|P_1(G)| \geq d$, a minimum d-transversal is obtained by taking d edges in $P_1(G)$. This is not necessarily true for a minimum d-blocker. In fact, consider the chain $C = \{x_1x_2, x_2x_3, x_3x_4\}$. We have $P_1(C) = \{x_1x_2, x_3x_4\}$, but clearly $P_1(C)$ is a blocker but not a 2-blocker for C.

We will now give some basic properties concerning d-transversals and d-blockers in a graph $G = (V, E)$. We shall always assume that $d \leq \nu(G)$.

Property 2.1. *In any graph G and for any $d \geq 1$, a d-blocker B is a d-transversal.*

Proof. If the removal of $B \subseteq E$ reduces the maximum cardinality of a matching by at least d, then every maximum matching will contain at least d edges of B: indeed if there were a maximum matching M in G with $|M \cap B| \leq d - 1$, then the matching $M \setminus B$ in $(V, E \setminus B)$ has cardinality $|M \setminus B| > \nu(G) - d$, contradicting the assumption that B is a d-blocker. □

Property 2.2. *In any graph $G = (V, E)$ a set T is a transversal if and only if it is a blocker.*

Proof. From Property 2.1, we just have to show that a transversal T is a blocker. By definition we have $M \cap T \neq \emptyset$ for every maximum matching M. It follows that after the removal of T, the matching number in G has decreased by at least one. □

Observe that in any graph G and for any $d \geq 1$, a d-transversal T is a blocker. In fact, a d-transversal T is also a transversal and hence from Property 2.2 we conclude that T is a blocker.

Remark 2.2. For $d \geq 2$, there are d-transversals which are not d-blockers. Fig. 1 shows in a graph $G = C_6$ (cycle on six vertices) a set $T \subseteq E$ (bold edges) which is a 2-transversal ($|M \cap T| \geq 2$ for every maximum matching). It is not a 2-blocker, since $\nu(G) = 3$ and in $G' = (V, E \setminus T)$ we have $\nu(G') = 2 > \nu(G) - 2 = 1$.