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Stability of the path–path Ramsey number

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ABSTRACT

Here we prove a stability version of a Ramsey-type Theorem for paths. Thus in any 2-coloring of the edges of the complete graph K_n we can either find a monochromatic path substantially longer than $2n/3$, or the coloring is close to the extremal coloring.

1. Introduction

The vertex-set and the edge-set of the graph G is denoted by $V(G)$ and $E(G)$. K_n is the complete graph on n vertices, $K_{r+1}(t)$ is the complete $(r+1)$-partite graph where each class contains t vertices and $K_2(t) = K(t, t)$ is the complete bipartite graph between two vertex classes of size t. We denote by (A, B, E) a bipartite graph $G = (V, E)$, where $V = A + B$, and $E \subset A \times B$. For a graph G and a subset U of its vertices, $G|_U$ is the restriction of G to U. The set of neighbors of $v \in V$ is $N(v)$. Hence the size of $N(v)$ is $|N(v)| = \deg(v) = \deg_G(v)$, the degree of v. The minimum degree is denoted by $\delta(G)$ and the maximum degree by $\Delta(G)$ in a graph G. When A, B are subsets of $V(G)$, we denote by $e(A, B)$ the number of edges of G with one endpoint in A and the other in B. In particular, we write $\deg(v, U) = e(\{v\}, U)$ for the number of edges from v to U. A graph G_n on n vertices is γ-*dense* if it has at least $\gamma \binom{n}{2}$ edges. A bipartite graph $G(k, l)$ is γ-dense if it contains at least γkl edges.

For graphs $G_1, G_2, \ldots, G_r$, the *Ramsey number* $R(G_1, G_2, \ldots, G_r)$ is the smallest positive integer n such that if the edges of a complete graph K_n are partitioned into r disjoint color classes giving r graphs $H_1, H_2, \ldots, H_r$, then at least one H_i $(1 \le i \le r)$ has a subgraph isomorphic to G_i. The existence of such a positive integer is guaranteed by Ramsey's classical result [12]. The number $R(G_1, G_2, \ldots, G_r)$ is called the Ramsey number for the graphs $G_1, G_2, \ldots, G_r$. There is very little known about $R(G_1, G_2, \ldots, G_r)$ even for very special graphs (see e.g. [4] or [11]). For $r = 2$ a theorem of Gerencsér and Gyárfás [3] states that

$$R(P_n, P_n) = \left\lfloor \frac{3n-2}{2} \right\rfloor. \tag{1}$$

In this paper we prove a stability version of this theorem. Since this is what we needed in a recent application [6], actually we prove the result in a slightly more general context; we work with 2-edge *multicolorings* (G_1, G_2) of a graph G. Here

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multicoloring means that the edges can receive more than one color, i.e. the graphs G_i are not necessarily edge disjoint. The subgraph colored with color *i only* is denoted by G_i^*, i.e.

$$G_1^* = G_1 \setminus G_2, \qquad G_2^* = G_2 \setminus G_1.$$

In order to state the theorem we need to define a relaxed version of the extremal coloring for (1).

Extremal Coloring (with parameter α): There exists a partition $V(G) = A \cup B$ such that

- $|A| \geq (2/3 - \alpha)|V(G)|$, $|B| \geq (1/3 - \alpha)|V(G)|$.
- The graph $G_1^*|_A$ is $(1-\alpha)$-dense and the bipartite graph $G_2^*|_{A\times B}$ is $(1-\alpha)$-dense. (Note that we have no restriction on the coloring inside the smaller set.)

Then the following stability version of the Gerencsér–Gyárfás Theorem claims that we can either find a monochromatic path substantially longer than $2n/3$, or the coloring is close to the extremal coloring.

Theorem 1.1. *For every $\alpha > 0$ there exists a positive real η ($0 < \eta \ll \alpha \ll 1$ where $\ll$ means sufficiently smaller) and a positive integer n_0 such that for every $n \geq n_0$ the following holds: if the edges of the complete graph K_n are 2-multicolored then we have one of the following two cases.*

- *Case* 1*: K_n contains a monochromatic path P of length at least $(\frac{2}{3} + \eta)n$.*
- *Case* 2*: This is an Extremal Coloring (EC) with parameter α.*

We remark that while for some classical density results the corresponding stability versions are well-known (see [1]), stability questions in Ramsey problems only emerged recently (see [5,7,10]).

2. Tools

Theorem 1.1 can also be proved from the Regularity Lemma [13], however, here we use a more elementary approach using only the Kővári–Sós–Turán bound [8]. This is part of a new direction to "de-regularize" some proofs, namely to replace the Regularity Lemma with more elementary classical extremal graph theoretic results such as the Kővári–Sós–Turán bound (see e.g. [9]).

Lemma 2.1 (*Theorem 3.1 on page 328 in [1]*). *There is an absolute constant $\beta > 0$ such that if $0 < \epsilon < 1/r$ and we have a graph G with*

$$|E(G)| \geq \left(1 - \frac{1}{r} + \epsilon\right) \frac{n^2}{2}$$

then G contains a $K_{r+1}(t)$, where

$$t = \left\lfloor \frac{\beta \log n}{r \log 1/\epsilon} \right\rfloor.$$

For $r = 1$ this is essentially the Kővári–Sós–Turán bound [8] and for general r this was proved by Bollobás, Erdős and Simonovits [2]. Here we will use the result only for $r = 1$.

3. Outline of the proof

We will need the following definition. Given a graph G and a positive integer k, we say that a subset W of the vertex set $V(G)$ is *k-well-connected* if for any two vertices $u, v \in W$ there are at least k internally vertex disjoint paths of length at most *three* connecting u and v in G (note that these paths might leave W). We will use this definition with $k = \eta n$, in this case we just say shortly that W is well-connected.

We will follow a similar outline as in applications of the Regularity Lemma. However, a regular pair will be replaced with a complete balanced bipartite graph $K(t, t)$ with $t \geq c \log n$ for some constant c (thus the size of the pair is somewhat smaller but this is still good enough for our purposes). Then a monochromatic connected matching in the reduced graph (the usual tool in these types of proofs using the Regularity Lemma) will be replaced with a set of vertex disjoint monochromatic complete balanced bipartite graphs $K_i(t_i, t_i)$, $1 \leq i \leq s$ with $t_i \geq c \log n$, $1 \leq i \leq s$ for some constant c. Moreover, these bipartite graphs are all contained in a set W that is well-connected in this color. Let us call a set of bipartite graphs like this a monochromatic well-connected complete balanced bipartite graph cover (we are covering vertices here). The size of this cover is the total number of vertices in the union of these complete bipartite graphs.

Then Theorem 1.1 will follow from the following lemma.

Lemma 3.1. *For every $\alpha > 0$ there exist a positive real η ($0 < \eta \ll \alpha \ll 1$ where $\ll$ means sufficiently smaller) and a positive integer n_0 such that for every $n \geq n_0$ the following holds: if the edges of the complete graph K_n are 2-multicolored then we have one of the following two cases.*

- *Case* 1*: K_n contains a monochromatic well-connected complete balanced bipartite graph cover of size at least $(\frac{2}{3} + 2\eta)n$.*
- *Case* 2*: This is an Extremal Coloring (EC) with parameter α.*