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The k-restricted edge connectivity of undirected Kautz graphs[☆]

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ABSTRACT

The k-restricted edge connectivity is a more refined network reliability index than edge connectivity. In this paper, we study the undirected Kautz graph $UK(d, n)$, an important model of networks, give an upper bound on the k-restricted edge connectivity of $UK(d, n)$ for some small k and determine the 4-restricted edge connectivity of $UK(2, n)$.

1. Introduction

For graph-theoretical terminology and notation not defined here we follow [4]. It is well known that the underlying topology of a processor interconnection network or a communications network can be modeled by a graph $G = (V, E)$, where the vertex set V corresponds to processors or switching elements and the edge set E corresponds to communication links. The Kautz graph [3,8,15] has been widely used in the design and analysis of interconnection networks. It can be defined as follows. The Kautz digraph, denoted by $K(d, n)$, where d, n are two given integers, $d \geq 1, n \geq 2$, has the vertex set $V = \{x_1x_2 \cdots x_n : x_i \in \{0, 1, \ldots, d\}, x_{i+1} \neq x_i, i = 1, 2, \ldots, n-1\}$, and the arc set A, where for every pair of vertices $x, y \in V$, if $x = x_1x_2 \cdots x_n$, then $(x, y) \in A \Leftrightarrow y = x_2x_3 \cdots x_{n+1}, x_{n+1} \in \{0, 1, \ldots, d\} - \{x_n\}$. Define $K(d, 1)$ as a complete digraph of order $(d + 1)$. Clearly, $K(d, n)$ is d-regular, i.e., for any $x \in V(K(d, n))$, both its out-degree $d^+(x)$ and its in-degree $d^-(x)$ are d. The undirected Kautz graph, denoted by $UK(d, n)$, is obtained from $K(d, n)$ by deleting the orientation of all arcs and keeping one edge of a pair of multiple edges.

As a more refined index than the edge connectivity, the k-restricted edge connectivity was proposed in [5,6]. A set of edges S in a connected graph G is called a k-restricted edge cut if $G - S$ is disconnected and every component of $G - S$ has at least k vertices. The k-restricted edge connectivity of G, denoted by $\lambda_k(G)$, is defined as the cardinality of a minimum k-restricted edge cut. A connected graph G is said to be λ_k-connected if $\lambda_k(G)$ exists. It is easy to see that if G is λ_k-connected for $k \geq 2$, then G is also λ_{k-1}-connected and $\lambda_{k-1}(G) \leq \lambda_k(G)$. In view of recent studies on k-restricted edge connectivity, it seems that the larger the $\lambda_k(G)$, the more reliable the network [11,10,16]. So, we expect $\lambda_k(G)$ to be as large as possible. Clearly, the optimization of $\lambda_k(G)$ requires an upper bound first. For subsets U and U' of $V(G)$, we denote by $[U, U']$ the set of edges with one end in U and the other in U'. For any positive integer k, let $\xi_k(G) = \min\{|[X, \overline{X}]| : X \subset V(G), |X| = k, G[X] \text{ is connected}\}$, where $\overline{X} = V(G) \setminus X$ and $G[X]$ is the subgraph of G induced by X. It has been shown that $\lambda_k(G) \leq \xi_k(G)$ holds for many graphs [5,12,19]. A connected graph G for which $\lambda_k(G) \leq \xi_k(G)$ holds is called a λ_k-optimal graph if $\lambda_k(G) = \xi_k(G)$. Sufficient conditions for graphs to be λ_k-connected or λ_k-optimal were given by several authors [5,6,12,19,1,2,9,18,20]. In particular, the λ_k-optimality of undirected Kautz graphs has attracted much attention recently. In 2004, Ou and Zhang [14] proved that

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for $n \geq 3$, $UK(2, n)$ is λ_2-optimal. In the same year, Fan [7] proved that for $n \geq 2$, $UK(3, n)$ is λ_2-optimal. In 2005, Wang and Lin [17] showed that for $d \geq 3$, $n \geq 2$, $UK(d, n)$ is λ_2-optimal. In 2007, Ou et al. [13] showed the following.

Proposition 1.1 *([13]). The undirected Kautz graph $UK(2, n)$ is λ_3-optimal when $n \geq 3$, that is, $\lambda_3(UK(2, n)) = \xi_3(UK(2, n)) = 6$.*

In this paper, we give an upper bound on $\lambda_k(UK(d, n))$ for some small k and show that $UK(2, n)$ is λ_4-optimal when $n \geq 4$.

2. Preliminaries

We begin with the basic structure and some useful properties of Kautz graphs. Clearly, $|V(UK(d, n))| = d^n + d^{n-1}$. A vertex $x = x_1x_2\cdots x_n$ of $UK(d, n)$ is called a binary vertex if $x_1 = x_3 = \cdots = a \neq b = x_2 = x_4 = \cdots$. Two binary vertices $x = x_1x_2\cdots x_n$, $y = y_1y_2\cdots y_n$ are said to be symmetric if $x_1 = y_2$ and $x_2 = y_1$. It is easy to see that x, y are symmetric binary vertices if and only if xyx is a directed 2-cycle in $K(d, n)$. Combining this with the definition of $UK(d, n)$, we have the following.

Lemma 2.1. *For $d \geq 2$, $n \geq 2$, $UK(d, n)$ has the minimum degree $2d - 1$, while the maximum degree is $2d$ for $n \geq 3$ and $2d - 1$ for $n = 2$. Furthermore, the degree $d(x)$ of the vertex x is $2d - 1$ if and only if x is a binary vertex.*

For $d \geq 2$, $n \geq 3$, a vertex $x = x_1x_2\cdots x_n$ of $UK(d, n)$ is called a trinary vertex if there exist distinct $a, b, c \in \{0, 1, \ldots, d\}$ such that $x_1 = x_4 = \cdots = a$, $x_2 = x_5 = \cdots = b$, $x_3 = x_6 = \cdots = c$. Two trinary vertices $x = x_1x_2\cdots x_n$, $y = y_1y_2\cdots y_n$ are said to be consistent if either $y_1 = x_2, y_2 = x_3, y_3 = x_1$ or $x_1 = y_2, x_2 = y_3, x_3 = y_1$.

For a vertex x of $UK(d, n)$, denote by $N^+(x)$ and $N^-(x)$ the out-neighbourhood and in-neighbourhood of x in $K(d, n)$, respectively. For three distinct vertices x, y, z of $UK(d, n)$, $xyzx$ is called a triangle of $UK(d, n)$ if xy, yz, zx are edges of $UK(d, n)$.

Lemma 2.2. *For $d \geq 2$, $n \geq 3$, $xyzx$ is a triangle of $UK(d, n)$ if and only if x, y, z are pairwise consistent trinary vertices.*

Proof. Suppose that $x = x_1x_2\cdots x_n$, $y = y_1y_2\cdots y_n$, $z = z_1z_2\cdots z_n$ are pairwise consistent trinary vertices of $UK(d, n)$. Since both x, y and x, z are consistent, without loss of generality, we can assume that $y_1 = x_2, y_2 = x_3, y_3 = x_1$ and $x_1 = z_2, x_2 = z_3, x_3 = z_1$. Combining this with the fact that x, y, z are trinary, it follows that $y_i = x_{i+1}, z_i = y_{i+1}, x_i = z_{i+1}$ for $i = 1, 2, \ldots, n-1$. By definition, $xyzx$ is a directed triangle of $K(d, n)$ and so a triangle of $UK(d, n)$.

Now suppose conversely that $xyzx$ is a triangle in $UK(d, n)$. Without loss of generality, assume $y \in N^+(x)$. Since any two vertices in $N^+(x)$ are not adjacent and y, z are adjacent, we have $z \in N^-(x)$, that is, $x \in N^+(z)$. Similarly, we have $y \in N^-(z)$. It follows that $xyzx$ is a directed triangle in $K(d, n)$. By the definition of $K(d, n)$, we have that x, y, z are pairwise consistent trinary vertices. □

For $d \geq 2$, $n \geq 3$, let $x = x_1x_2\cdots x_n$ be a vertex of $UK(d, n)$, and let $x^{(1)} = x_1x_2\cdots x_{n-1}$, $x^{(2)} = x_2x_3\cdots x_n$. Then $x^{(1)}, x^{(2)}$ are vertices of $UK(d, n-1)$ and $x^{(1)}x^{(2)}$ is an edge of $UK(d, n-1)$. For a path $P = u_0u_1u_2\cdots u_k$ of $UK(d, n)$, denote by $G(P)$ the subgraph of $UK(d, n-1)$ with vertex set $\cup_{i=0}^{k}\{u_i^{(1)}, u_i^{(2)}\}$ and edge set $\cup_{i=0}^{k}\{u_i^{(1)}u_i^{(2)}\}$. Since u_i and u_{i+1} are adjacent, we have $\{u_i^{(1)}, u_i^{(2)}\} \cap \{u_{i+1}^{(1)}, u_{i+1}^{(2)}\} \neq \emptyset$, $i = 0, 1, \ldots, k-1$.

Lemma 2.3. *For $d \geq 2$, $n \geq 3$, let $P = u_0u_1\cdots u_k$ be a path of $UK(d, n)$. Then $G(P)$ is a connected subgraph of $UK(d, n-1)$ with at most $k+1$ edges.*

Proof. By induction on the length k of P. This is clearly true for $k = 0$. Suppose, then, that the lemma holds for any path of $UK(d, n)$ with length less than k, where $k \geq 1$. Since $\{u_{k-1}^{(1)}, u_{k-1}^{(2)}\} \cap \{u_k^{(1)}, u_k^{(2)}\} \neq \emptyset$, we can assume, without loss of generality, $u_{k-1}^{(2)} = u_k^{(1)}$. Let $P' = u_0u_1\cdots u_{k-1}$. Then $V(G(P)) = V(G(P')) \cup \{u_k^{(2)}\}$ and $E(G(P)) = E(G(P')) \cup \{u_{k-1}^{(2)}u_k^{(2)}\}$. By the induction hypothesis, $G(P')$ is a connected subgraph of $UK(d, n-1)$ with at most k edges. It follows that $G(P)$ is a connected subgraph of $UK(d, n-1)$ with at most $k+1$ edges. The proof is complete. □

Lemma 2.4. (a) *For $d \geq 2$, $n \geq 2$, let x, y be two binary vertices of $UK(d, n)$. If they are not symmetric, then the distance between them is at least $n-1$, that is, $d(x, y) \geq n-1$.*

(b) *For $d \geq 2$, $n \geq 3$, let x, y be two trinary vertices of $UK(d, n)$. If they are not consistent, then $d(x, y) \geq n-1$.*

Proof. First, we prove Part (a). Clearly, Part (a) is true for $n = 2$. Suppose Part (a) fails and we take the minimal $n \geq 3$ for which there are two non-symmetric binary vertices x, y in $UK(d, n)$ with $d(x, y) = m \leq n-2$. Since x, y are not symmetric, we have $m \geq 2$. Let $P = u_0u_1\cdots u_m$ be a shortest path from x to y, where $u_0 = x$, $u_m = y$, and let $Q = u_1\cdots u_{m-1}$. Since $u_0u_1, u_{m-1}u_m \in E(UK(d, n))$, it follows that $\{u_i^{(1)}, u_i^{(2)}\} \cap \{u_{i+1}^{(1)}, u_{i+1}^{(2)}\} \neq \emptyset$, $i = 0, m-1$. So, we can assume, without loss of generality, that $u_0^{(2)} \in \{u_1^{(1)}, u_1^{(2)}\}$, $u_m^{(1)} \in \{u_{m-1}^{(1)}, u_{m-1}^{(2)}\}$, which implies that $u_0^{(2)}, u_m^{(1)}$ are two vertices in $G(Q)$. By Lemma 2.3, $G(Q)$ is a connected subgraph of $UK(d, n-1)$ with at most $m-1 \leq n-3$ edges. It follows that the distance between $u_0^{(2)}$ and $u_m^{(1)}$ is at most $n-3$. On the other hand, since u_0, u_m are two non-symmetric binary vertices in $UK(d, n)$, it follows that $u_0^{(2)}, u_m^{(1)}$ are two non-symmetric binary vertices in $UK(d, n-1)$. By the minimality of n, $d(u_0^{(2)}, u_m^{(1)}) \geq (n-1) - 1 = n-2$, a contradiction completing the proof of Part (a). Similarly, we can prove Part (b). □