Discrete Mathematics 309 (2009) 2010–2014

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

On zero-sum partitions and anti-magic trees

Gil Kaplan *, Arieh Lev, Yehuda Roditty

School of Computer Sciences, The Academic College of Tel-Aviv-Yaffo, 2 Rabenu Yeruham Street, Tel-Aviv, 61083, Israel

ARTICLE INFO

Article history:
Received 22 October 2007
Received in revised form 27 March 2008
Accepted 3 April 2008
Available online 19 May 2008

Keywords:
Zero-sum partitions
Anti-magic graphs

ABSTRACT

We study zero-sum partitions of subsets in abelian groups, and apply the results to the study of anti-magic trees. Extension to the nonabelian case is also given.

1. Introduction

In this paper we study zero-sum partitions of subsets in groups, and apply the results to the study of anti-magic trees. Extension to the nonabelian case is also given. The notation used is standard, and we generally follow the notation in [8] and [9]. Abelian groups will be written additively and nonabelian groups will be written multiplicatively, but the identity element will be always denoted by 0.

We start with some basic definitions.

Definition 1. Let G be an abelian group and let A be a finite subset of $G-\{0\}$, with $|A|=n$. We shall say that A has the *zero-sum-partition* property (ZSP-property) if for every partition $n=r_1+r_2+\cdots+r_t$ of n, with $r_i\geq 2$ for $1\leq i\leq t$, there is a partition of A into pairwise disjoint subsets $A_1,A_2,\ldots,A_t$, such that $|A_i|=r_i$ and $\sum_{a\in A_i}a=0$ for $1\leq i\leq t$. In case that G is finite, we shall say that G has the ZSP-property if $A=G-\{0\}$ has the ZSP-property.

Definition 2. A 2-*tree* T is a rooted tree, where each vertex $v\in V(T)$ which is not a leaf has at least two children.

Part (iii) of the following definition is the notion of anti-magic graphs that was introduced by Hartsfield and Ringel [5, pp. 109]. In our terminology, this notion is a special case of part (i) of the definition.

Definition 3. Let $H=(V,E)$ be a graph, where $|V|=n$, $|E|=m$. Let G be an abelian group and let A be a finite subset of $G-\{0\}$ with $|A|=m$. An *A-labeling* of H is a one-to-one mapping $l:E(H)\to A$. Given an A-labeling of H, the *weight* of a vertex $v\in V(G)$ is $w(v)=\sum_{uv\in E(H)}l(uv)$.

(i) We shall say that H is *A-anti-magic* if there is an A-labeling of H such that the weights $\{w(v)|v\in V(H)\}$ are all distinct.
(ii) In case that G is finite, we shall say that H is *G-anti-magic* if H is $(G-\{0\})$-anti-magic.
(iii) We shall say that H is *anti-magic* if H is A-anti-magic, where $A=\{1,2,\ldots,m\}\subset G=(\mathbf{Z},+)$.

Hartsfield and Ringel conjectured that every connected graph but K_2 is anti-magic. This conjecture, as well as the conjecture (a particular case) that every tree but K_2 is anti-magic, is still open. Recent results on anti-magic graphs may be found in [1]. We mention also the related problem of product anti-magic graphs, where the labeling is still by the elements

* Corresponding author.
E-mail address: gilk@mta.ac.il (G. Kaplan).

0012-365X/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2008.04.012

of the set $\{1, 2, \ldots, n\}$, but the weight of a vertex is the *product* of the labels of its neighboring edges. Recent results on product anti-magic graphs may be found in [2,6,7]. Definition 3 above extends the notions of anti-magic graphs and product anti-magic graphs to A-anti-magic graphs, where A is a subset of any abelian group.

We review now our results, starting with the following theorem.

Theorem A. *Let $n = r_1+r_2+\cdots+r_t$ be a partition of the positive integer n, where $r_i \geq 2$ for $i = 1, 2, \ldots, t$. Let $A = \{1, 2, \ldots, n\}$. Then the set A can be partitioned into pairwise disjoint subsets $A_1, A_2, \ldots, A_t$ such that for every $1 \leq i \leq t$, $|A_i| = r_i$ with $\sum_{a\in A_i} a \equiv 0 \pmod{n+1}$ if n is even and $\sum_{a\in A_i} a \equiv 0 \pmod{n}$ if n is odd.*

Let $G = (\mathbf{Z}_n, +)$ be the additive group of integers modulo n. Using the result of Theorem A, we are able to determine whether G has the ZSP-property, to prove that every 2-tree is anti-magic, and to classify the G-anti-magic 2-trees, as given in Theorems B and C.

Theorem B. *Let $n \geq 2$ be a natural number and let $G = (\mathbf{Z}_n, +)$, the additive group of integers modulo n. Then G has the ZSP-property if and only if n is odd.*

Theorem C. *Let $T = (V, E),\ |V| = n \geq 2$, be a 2-tree. Then*

1. *T is anti-magic.*
2. *Let $G = (\mathbf{Z}_n, +)$ be the additive group of integers modulo n. Then T is G-anti-magic if and only if n is odd.*

Actually, the proof of Theorem C shows that the following simple proposition holds:

Proposition D. *Let G be a finite abelian group which has the ZSP-property. Then every 2-tree on $|G|$ vertices is G-anti-magic.*

The above proposition is very useful in studying G-anti-magic 2-trees, and will be used in what follows.

We turn now to the case where G is any finite abelian group. Recall that an involution in G is an element of G of order 2. We conjecture the following:

Conjecture. *Let T be a tree (not necessarily a 2-tree) and let G be a finite abelian group, where $|G| = |V(T)|$. Then*

1. *G has the ZSP-property if and only if either G is of odd order or G contains exactly 3 involutions.*
2. *T is G-anti-magic if and only if G is not a group with a unique involution.*

One direction of the conjecture is not hard and is given in the following Theorem.

Theorem E. *Let G be a finite abelian group of even order n, and assume that the number of involutions in G is different from 3. Then*

1. *G does not have the ZSP-property.*
2. *If G has a unique involution, then every tree on n vertices is not G-anti-magic.*

We do not have the answer for the opposite direction, however, we shall give a proof of a particular case. Recall first, that a finite group is *elementary abelian* if it is the direct product of cyclic groups of order p, where p is a prime.

Theorem F. *Let G be an elementary abelian group of order $n = p^k$, where p is a prime congruent to $1 \pmod 3$. Then*

1. *G has the ZSP-property.*
2. *Every 2-tree on n vertices is G-anti-magic.*

Using a similar proof to that of Theorem F we may extend the result of the theorem to some *nonabelian* groups. In order to do so we need to extend Definitions 1 and 3 to nonabelian groups. The extension is as follows:

1. In Definition 1 we allow the group to be nonabelian, but require that for each subset A_i in the partition of A ($1 \leq i \leq t$), there will exist a permutation of the elements of A_i: $A_i = \{a_{i_1}, a_{i_2}, \ldots, a_{i_{s(i)}}\}$ ($s(i) = |A_i|$), such that $a_{i_1}a_{i_2}\cdots a_{i_{s(i)}} = 0$.
2. Similarly, in Definition 3 we shall require that for $v_i \in V(T)$ ($1 \leq i \leq n$), there will exist a permutation of the labels of the edges adjacent to v_i: $\{g_{i_1}, g_{i_2}, \ldots, g_{i_{s(i)}}\}$ ($s(i) = \deg(v_i)$), such that the corresponding weights $w(v_i) = g_{i_1}g_{i_2}\cdots g_{i,s(i)}$ are all distinct.

For the following corollary, recall that a Frobenius group is a group having a subgroup H (called the Frobenius complement) such that $H\bigcap x^{-1}Hx = \{0\}$ for every $x \in G - H$. It is known (see [8, 8.5.5]) that every Frobenius group G has a normal subgroup N (called the Frobenius kernel) such that $H\bigcap N = \{0\}$ and $G = HN$.

Corollary G. *Let G be a Frobenius group of odd order n. Suppose that a complement of G and the kernel of G are either a cyclic group of order congruent to $1 \pmod 3$ or an elementary abelian r-group, where r is a prime congruent to $1 \pmod 3$. Then items 1 and 2 of Theorem F hold for the group G. In particular, items 1and 2 of Theorem F hold if G is a Frobenius group of order $n = pq$, where p and q are distinct odd primes which are both congruent to $1 \pmod 3$.*