Discrete Mathematics 309 (2009) 2015–2019

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Hamilton cycles in claw-heavy graphs

Bing Chen [a], Shenggui Zhang [b,*], Shengning Qiao [b]

[a] Department of Applied Mathematics, Xi'an University of Technology, Xi'an, Shaanxi 710048, PR China

[b] Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, PR China

ARTICLE INFO

Article history:
Received 9 May 2007
Received in revised form 3 April 2008
Accepted 3 April 2008
Available online 20 May 2008

Keywords:
Hamilton cycle
2-heavy graph
Claw-heavy graph

ABSTRACT

A graph G on $n \geq 3$ vertices is called claw-heavy if every induced claw ($K_{1,3}$) of G has a pair of nonadjacent vertices such that their degree sum is at least n. In this paper we show that a claw-heavy graph G has a Hamilton cycle if we impose certain additional conditions on G involving numbers of common neighbors of some specific pair of nonadjacent vertices, or forbidden induced subgraphs. Our results extend two previous theorems of Broersma, Ryjáček and Schiermeyer [H.J. Broersma, Z. Ryjáček, I. Schiermeyer, Dirac's minimum degree condition restricted to claws, Discrete Math. 167–168 (1997) 155–166], on the existence of Hamilton cycles in 2-heavy graphs.

1. Terminology and notation

We use Bondy and Murty [4] for terminology and notation not defined here and consider finite simple graphs only.

Let G be a graph on n vertices. For a vertex $v \in V(G)$, its *neighbor*, denoted by $N_G(v)$, and its *degree*, denoted by $d_G(v)$, are defined as the set and the number of vertices in G that are adjacent to v, respectively. By $\delta(G)$ we mean the minimum degree of the vertices of G. We often simply write $N(v)$, $d(v)$ and δ instead of $N_G(v)$, $d_G(v)$ and $\delta(G)$ if there is no possibility of confusion. If $S \subseteq V(G)$, then $\langle S \rangle$ denotes the subgraph of G induced by S. A graph H is an *induced subgraph* of G if $H = \langle S \rangle$ for some $S \subseteq V(G)$. An induced subgraph of G with vertex set $\{u, v, w, x\}$ and edge set $\{uv, uw, ux\}$ is called a *claw* of G, with *center* u and *end vertices* v, w, x. An induced subgraph of G isomorphic to $K_{1,3}$ with one additional edge is called a *modified claw*. A vertex v of G is called *heavy* if $d(v) \geq n/2$. A claw of G is called 1-*heavy* if at least one of its end vertices is heavy, and is called 2-*heavy* if at least two of its end vertices are heavy. A graph is called 1-*heavy* (2-*heavy*) if all its induced claws are 1-heavy (2-heavy, respectively).

If H is a graph, we say that a graph G is H-*free* if G contains no copy of H as an induced subgraph. Instead of $K_{1,3}$-free, we use the more common term *claw-free*. Note that every claw-free graph is 2-heavy, and that every 2-heavy graph is 1-heavy.

We use D (of *deer*), H (of *hourglass*) and P_7 (*of a path on 7 vertices*) to denote the graphs of Fig. 1.

2. Main results

Degree conditions and forbidden subgraph conditions are two important types of sufficiency conditions for the existence of Hamilton cycles in graphs. The following are two examples of these two types of conditions, respectively.

Theorem 1 (*Dirac [8]*). *Let G be a graph on $n \geq 3$ vertices with $\delta(G) \geq n/2$. Then G is hamiltonian.*

* Corresponding author.
E-mail address: sgzhang@nwpu.edu.cn (S. Zhang).

0012-365X/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2008.04.013

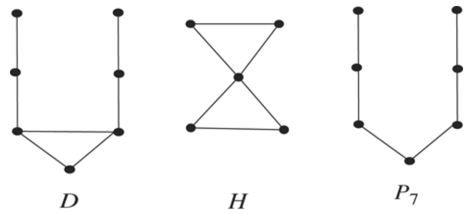

Fig. 1. Graphs D, H and P_7.

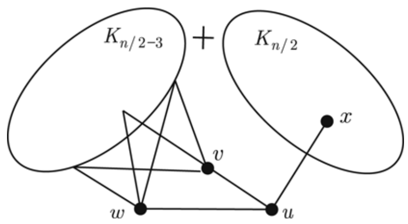

Fig. 2. The graph G.

Theorem 2 (*Shi [13]*). *Let G be a 2-connected graph on $n \geq 3$ vertices. If G is claw-free and $|N(u) \cap N(v)| \geq 2$ for every pair of vertices u, v with $d(u, v) = 2$, then G is hamiltonian.*

Combining these two types of conditions, Broersma et al. [5] gave a common generalization of Theorems 1 and 2.

Theorem 3 (*Broersma et al. [5]*). *Let G be a 2-connected graph on $n \geq 3$ vertices. If G is 2-heavy and $|N(u) \cap N(v)| \geq 2$ for every pair of vertices u, v in a modified claw with $d(u, v) = 2$ and $\max\{d(u), d(v)\} < n/2$, then G is hamiltonian.*

Theorem 3 also generalizes several other results, due to Bedrossian et al. [1], Fan [9] and Ore [12], on the existence of Hamilton cycles in graphs.

A graph G on n vertices is called *claw-heavy* if every induced claw has a pair of nonadjacent vertices u and v satisfying $d(u) + d(v) \geq n$. Fujisawa and Yamashita [11] introduced this notion and gave a new sufficiency condition for the existence of Hamilton cycles in graphs.

It is clear that every claw-heavy graph is 1-heavy, but not necessarily 2-heavy. Broersma et al. [5] showed that one cannot relax 2-heavy to 1-heavy in Theorem 3. Our first objective in this paper is to prove that we can relax 2-heavy to claw-heavy in Theorem 3.

Theorem 4. *Let G be a 2-connected graph on $n \geq 3$ vertices. If G is claw-heavy and $|N(u) \cap N(v)| \geq 2$ for every pair of vertices u, v in a modified claw with $d(u, v) = 2$ and $\max\{d(u), d(v)\} < n/2$, then G is hamiltonian.*

For claw-free graphs, the following two results on the existence of Hamilton cycles are also known.

Theorem 5 (*Broersma and Veldman [6]*). *Let G be a 2-connected graph. If G is claw-free, P_7-free and D-free, then G is hamiltonian.*

Theorem 6 (*Faudree et al. [10]*). *Let G be a 2-connected graph. If G is claw-free, P_7-free and H-free, then G is hamiltonian.*

Broersma et al. [5] extended Theorems 5 and 6 to the class of 2-heavy graphs.

Theorem 7 (*Broersma et al. [5]*). *Let G be a 2-connected graph. If G is 2-heavy, and moreover, P_7-free and D-free, or P_7-free and H-free, then G is hamiltonian.*

Broersma et al. [5] also showed that one cannot relax 2-heavy to 1-heavy in Theorem 7. Here, as in Theorem 4, we prove that we can relax 2-heavy to claw-heavy in Theorem 7.

Theorem 8. *Let G be a 2-connected graph on $n \geq 3$ vertices. If G is claw-heavy, and moreover, P_7-free and D-free, or P_7-free and H-free, then G is hamiltonian.*

Remark 1. The graph in Fig. 2 shows our results in Theorems 4 and 8 does strengthen those in Theorems 3 and 7. Let $n \geq 10$ be an even integer and $K_{n/2} + K_{n/2-3}$ denote the join of two complete graphs $K_{n/2}$ and $K_{n/2-3}$. Choose a vertex $x \in V(K_{n/2})$ and construct a graph G with $V(G) = V(K_{n/2}+K_{n/2-3}) \cup \{u, v, w\}$ and $E(G) = E(K_{n/2}+K_{n/2-3}) \cup \{uv, uw, ux\} \cup \{vy, wy | y \in V(K_{n/2-3})\}$. It is easy to see that G is a hamiltonian graph satisfying the conditions of Theorems 4 and 8, but not the conditions of Theorems 3 and 7.

We postpone the proofs of Theorems 4 and 8 to the next section.