Discrete Mathematics 309 (2009) 2119–2133

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Netlike partial cubes III. The median cycle property

Norbert Polat

I.A.E., Université Jean Moulin (Lyon 3), 6 cours Albert Thomas, 69355 Lyon Cedex 08, France

ARTICLE INFO

Article history:
Received 14 March 2007
Received in revised form 10 April 2008
Accepted 14 April 2008
Available online 27 May 2008

Keywords:
Hypercube
Partial cube
Netlike partial cube
Median graph
Even cycle
Benzenoid graph
Cellular bipartite graph
Geodesic convexity
Gated set
Gated amalgamation
Median cycle
Expansion
Pash–Peano space

ABSTRACT

A graph G has the *Median Cycle Property* (MCP) if every triple (u_0, u_1, u_2) of vertices of G admits a unique median or a unique median cycle, that is a gated cycle C of G such that for all $i, j, k \in \{0, 1, 2\}$, if x_i is the gate of u_i in C, then: $\{x_i, x_j\} \subseteq I_G(u_i, u_j)$ if $i \neq j$, and $d_G(x_i, x_j) < d_G(x_i, x_k) + d_G(x_k, x_j)$. We prove that a netlike partial cube has the MCP if and only if it contains no triple of convex cycles pairwise having an edge in common and intersecting in a single vertex. Moreover a finite netlike partial cube G has the MCP if and only if G can be obtained from a set of even cycles and hypercubes by successive gated amalgamations, and equivalently, if and only if G can be obtained from K_1 by a sequence of special expansions. We also show that the geodesic interval space of a netlike partial cube having the MCP is a Pash–Peano space (i.e. a closed join space).

1. Introduction

In [3] Bandelt and Chepoi investigated the cellular bipartite graphs, that is the graphs that can be obtained from single edges and even cycles by successive gated amalgamations. They concluded their paper by suggesting that the class of cellular bipartite graphs should be generalized in order to incorporate the median graphs by considering the graphs obtained from even cycles and hypercubes via successive gated amalgamations.

The class of netlike partial cubes, that we introduced in the first paper [15] of this series, is closed under gated amalgamations and contains in particular the classes of median graphs, cellular bipartite graphs and benzenoid graphs. More precisely, cellular bipartite graphs and benzenoid graphs are instances of the special netlike partial cubes that we called linear partial cubes. So the graphs that Bandelt and Chepoi proposed to investigate are special netlike partial cubes.

Now Bandelt and Chepoi [3] proved in particular that the cellular bipartite graphs have, what we call in this paper, the Median Cycle Property (MCP for short), i.e. the property that any triple (u_0, u_1, u_2) of vertices of a cellular bipartite graph G admits a unique median or a unique median cycle, that is a particular gated cycle whose vertex set is contained in the union of intervals $I_G(u_0, u_1) \cup I_G(u_1, u_2) \cup I_G(u_2, u_0)$. We show in this paper (Theorem 3.5) that the graphs obtained from even cycles and hypercubes via successive gated amalgamations are precisely the finite netlike partial cubes which have the MCP, the cellular bipartite graphs being the finite linear partial cubes having the MCP.

We prove that, just like the geodesic interval spaces of a median graph and of a cellular bipartite graph, the interval space of a netlike partial cube having the MCP is a Pash–Peano space or, in other words, a (closed) join space. An abstract

E-mail address: norbert.polat@univ-lyon3.fr.

doi:10.1016/j.disc.2008.04.033

join space is composed of a set and a join operator for which the join of two points is interpreted as the segment between these two points, and satisfying certain axioms that are similar to the basic properties of line segments in Euclidean geometry. For a graph, the segment between two vertices is the geodesic interval between these vertices. This property of the geodesic interval space of a netlike partial cube having the MCP entails that the induced geodesic convexity has the Join-Hull Commutativity Property and the separation property S4.

Since Mulder [13] and Chepoi [7] introduced the expansion procedure for median graphs and partial cubes, different kinds of finite partial cubes have already been constructed from K_1 by sequences of special expansions (see [11]). In Section 6 we show that, if one can obtain all netlike partial cubes by Chepoi's theorem, not all graphs in the middle of expansion are netlike. More precisely, there exist infinitely many finite netlike partial cubes, in particular some benzenoid graphs, which are not the expansion of any netlike partial cubes. However we proved that there exists a particular kind of expansion that enables constructing all finite netlike partial cubes having the MCP from K_1.

Several results of Section 3 are generalizations of some results of the paper by Bandelt and Chepoi [3], and parts of their proofs may be analogous to the proofs of the corresponding results in [3]. So, in order to avoid unnecessary repetitions, we will as far as possible refer the reader to the relevant proofs in [3].

2. Preliminaries

2.1. Graphs

The graphs we consider are undirected, without loops or multiple edges, and may be finite or infinite. Let G be a graph. If $x \in V(G)$, the set $N_G(x) := \{y \in V(G) : xy \in E(G)\}$ is the *neighborhood* of x in G, $N_G[x] := \{x\} \cup N_G(x)$ is the *closed neighborhood* of x in G and $\delta_G(x) := |N_G(x)|$ is the *degree* of x in G. For a set X of vertices of G we put $N_G[X] := \bigcup_{x \in X} N_G[x]$ and $N_G(X) := N_G[X] - X$, we denote by $G[X]$ the subgraph of G induced by X, and we set $G - X := G[V(G) - X]$.

A *path* $P = \langle x_0, \ldots, x_n \rangle$ is a graph with $V(P) = \{x_0, \ldots, x_n\}$, $x_i \neq x_j$ if $i \neq j$, and $E(P) = \{x_i x_{i+1} : 0 \leq i < n\}$. A path $P = \langle x_0, \ldots, x_n \rangle$ is called an (x_0, x_n)*-path*, x_0 and x_n are its *endvertices*, while the other vertices are called its *inner* vertices, $n = |E(P)|$ is the *length* of P. If x and y are two vertices of a path P, then we denote by $P[x, y]$ the subpath of P whose endvertices are x and y.

A *cycle* C with $V(C) = \{x_1, \ldots, x_n\}$, $x_i \neq x_j$ if $i \neq j$, and $E(C) = \{x_i x_{i+1} : 1 \leq i < n\} \cup \{x_n x_1\}$, will be denoted by $\langle x_1, \ldots, x_n, x_1 \rangle$. The non-negative integer $n = |E(C)|$ is the *length* of C, and a cycle of length n is called an n*-cycle* and is often denoted by C_n.

Let G be a connected graph. The usual *distance* between two vertices x and y, that is, the length of an (x, y)*-geodesic* (=shortest (x, y)-path) in G, is denoted by $d_G(x, y)$. A connected subgraph H of G is *isometric* in G if $d_H(x, y) = d_G(x, y)$ for all vertices x and y of H. The *(geodesic) interval* $I_G(x, y)$ between two vertices x and y of G is the set of vertices of all (x, y)-geodesics in G.

2.2. Convexities

A *convexity* on a set X is an algebraic closure system $\mathcal{C}$ on X. The elements of $\mathcal{C}$ are the *convex sets* and the pair $(X, \mathcal{C})$ is called a *convex structure*. See van de Vel [18] for a detailed study of abstract convex structures. Several kinds of graph convexities, that is convexities on the vertex set of a graph G, have already been investigated. We will principally work with the *geodesic convexity*, that is the convexity on $V(G)$ which is induced by the geodesic interval operator I_G. In this convexity, a subset C of $V(G)$ is convex provided it contains the geodesic interval $I_G(x, y)$ for all $x, y \in C$. The *convex hull* $co_G(A)$ of a subset A of $V(G)$ is the smallest convex set which contains A. The convex hull of a finite set is called a *polytope*. A subset H of $V(G)$ is a *half-space* if H and $V(G) - H$ are convex. We will denote by $\mathcal{I}_G$ the pre-hull operator of the geodesic convex structure of G, i.e. the self-map of $\mathcal{P}(V(G))$ such that $\mathcal{I}_G(A) := \bigcup_{x,y \in A} I_G(x, y)$ for each $A \subseteq V(G)$. The convex hull of a set $A \subseteq V(G)$ is then $co_G(A) = \bigcup_{n \in \mathbb{N}} \mathcal{I}_G^n(A)$. Furthermore we will say that a subgraph of a graph G is *convex* if its vertex set is convex, and by the *convex hull* $co_G(H)$ of a subgraph H of G we will mean the smallest convex subgraph of G containing H as a subgraph, that is

$$co_G(H) := G[co_G(V(H))].$$

2.3. Netlike partial cubes

First we will recall some properties of *partial cubes*, that is of isometric subgraphs of hypercubes. Partial cubes are particular connected bipartite graphs.

For an edge ab of a graph G, let

$$W_{ab}^G := \{x \in V(G) : d_G(a, x) < d_G(b, x)\},$$
$$U_{ab}^G := N_G(W_{ba}^G).$$

If no confusion is likely, we will simply denote W_{ab}^G and U_{ab}^G by W_{ab} and U_{ab}, respectively. Note that the sets W_{ab} and W_{ba} are disjoint and that $V(G) = W_{ab} \cup W_{ba}$ if G is bipartite and connected.