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Nordhaus–Gaddum inequalities for the fractional and circular chromatic numbers

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ABSTRACT

For a graph G on n vertices with chromatic number $\chi(G)$, the Nordhaus–Gaddum inequalities state that $\lceil 2\sqrt{n}\rceil \leq \chi(G)+\chi(\overline{G}) \leq n+1$, and $n \leq \chi(G)\cdot\chi(\overline{G}) \leq \left\lfloor \left(\frac{n+1}{2}\right)^2 \right\rfloor$. Much analysis has been done to derive similar inequalities for other graph parameters, all of which are integer-valued. We determine here the optimal Nordhaus–Gaddum inequalities for the circular chromatic number and the fractional chromatic number, the first examples of Nordhaus–Gaddum inequalities where the graph parameters are *rational-valued*.

1. Introduction

In [19], Nordhaus and Gaddum determined bounds for the sum and product of the chromatic numbers of a graph and its complement.

Theorem 1.1 ([19]). *Let G be a graph on n vertices. Then,*

$$\lceil 2\sqrt{n}\rceil \leq \chi(G)+\chi(\overline{G}) \leq n+1,$$
$$n \leq \chi(G)\cdot\chi(\overline{G}) \leq \left\lfloor \left(\frac{n+1}{2}\right)^2 \right\rfloor.$$

Nordhaus and Gaddum also showed that these bounds are optimal by finding examples of graphs for which equality is reached. Since then, various papers have been published on determining optimal bounds for $\pi(G)+\pi(\overline{G})$ and $\pi(G)\cdot\pi(\overline{G})$, for other graph parameters π. In the literature, these results are known as *Nordhaus–Gaddum inequalities*.

We say that the function $f(n)$ is an *optimal lower bound* for $\pi(G)+\pi(\overline{G})$ if for every integer n, $f(n) \leq \pi(G)+\pi(\overline{G})$ for any graph G on n vertices, and the value $f(n)$ cannot be replaced by any larger real number (an *optimal upper bound* is defined analogously). Since there are only finitely many graphs on n vertices, the optimal bound $f(n)$ is simply the minimum value of $\pi(G)+\pi(\overline{G})$ over all possible graphs G on n vertices. Thus, for every n there must be at least one graph G with n vertices for which equality is attained. As a specific example, $f(n)=\lceil 2\sqrt{n}\rceil$ is *the* optimal lower bound for $\chi(G)+\chi(\overline{G})$, as shown in [19]. In some papers, it is written that $2\sqrt{n} \leq \pi(G)+\pi(\overline{G})$ is the optimal lower bound; by our definition, that will not be the case.

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Nordhaus–Gaddum inequalities have been established for numerous other graph parameters, such as the independence and edge-independence number [3,8], list-colouring number [7,10], diameter, girth, circumference, and edge-covering number [25], connectivity and edge-connectivity number [6], achromatic and pseudoachromatic number [1,26], and arboricity [18,23]. In some cases, bounds are found, yet it is unknown if they are optimal. A survey of known theorems (pre-1971) is given in [2]. As an example, two such results are as follows:

Let $\alpha_1(G)$ be the edge-independence number of G. Then, it is shown [3] that

$$\left\lfloor \frac{n}{2} \right\rfloor \leq \alpha_1(G) + \alpha_1(\overline{G}) \leq 2 \cdot \left\lfloor \frac{n}{2} \right\rfloor,$$
$$0 \leq \alpha_1(G) \cdot \alpha_1(\overline{G}) \leq \left\lfloor \frac{n}{2} \right\rfloor^2.$$

Let $\beta_1(G)$ be the edge-covering number of G. Then, it is shown [25] that

$$2 \cdot \left\lceil \frac{n}{2} \right\rceil \leq \beta_1(G) + \beta_1(\overline{G}) \leq 2n - 2 - \left\lfloor \frac{n}{2} \right\rfloor,$$
$$\left\lfloor \frac{n}{2} \right\rfloor^2 \leq \beta_1(G) \cdot \beta_1(\overline{G}) \leq \frac{n(n-1)}{2}.$$

In all of the known examples, the parameter $\pi(G)$ is *integer*-valued. In this paper, we provide the first instances of Nordhaus–Gaddum inequalities where the parameters are *rational*-valued, and our optimal bounds are non-integers. We will determine the optimal bounds for $\pi(G)+\pi(\overline{G})$ and $\pi(G)\cdot\pi(\overline{G})$, when $\pi(G)$ is the *fractional chromatic number* of G (denoted by $\chi_f(G)$), and when $\pi(G)$ is the *circular chromatic number* of G (denoted by $\chi_c(G)$). We will establish these Nordhaus–Gaddum inequalities using a generalization of the well-known Ramsey function, motivated by a technique in [3].

2. Definitions

For any graph G, the *clique number* $\omega(G)$ is the cardinality of the largest clique in G, and the *independence number* $\alpha(G)$ is the cardinality of the largest independent set in G. The *chromatic number* of a graph, $\chi(G)$, is the smallest size of a cover of the vertices of G by independent sets. We can alternatively define $\chi(G)$ using an integer program (IP) [5]. Let M denote the vertex-independent set incidence matrix of G. The rows are indexed by the vertices $\{v_1, v_2, \dots, v_n\}$, and the columns are indexed by the independent subsets of the vertices, $\{I_1, I_2, \dots, I_m\}$. The (i, j) entry of M is 1 when $v_i \in I_j$, and is 0 otherwise. Then $\chi(G) = \min \mathbf{1}^{\mathrm{T}}\mathbf{x}$, where $M\mathbf{x} \geq \mathbf{1}$, $\mathbf{x} \geq \mathbf{0}$, and $\mathbf{x} \in \mathbb{Z}^m$ (where $\mathbf{1}$ denotes the m by 1 vector of all 1's).

Definition 2.1 (*[21]*). Let M be the vertex-independent set incidence matrix of G. Then, the *fractional chromatic number* $\chi_f(G)$ is the relaxation of the integer program for $\chi(G)$ into a linear program:

$$\chi_f(G) = \min \mathbf{1}^{\mathrm{T}}\mathbf{x}, \quad \text{where } M\mathbf{x} \geq \mathbf{1}, \mathbf{x} \geq \mathbf{0}, \text{ and } \mathbf{x} \in \mathbb{R}^m.$$

Note that by definition, $\chi_f(G) \leq \chi(G)$, for all graphs G. By taking the integer program of a graph parameter and relaxing the IP into a linear program, we may define a corresponding *fractional* analogue (see [21]). This enables us to define parameters such as the fractional clique number, fractional domination number, fractional matching number, among many others. It is known [21] that each of these fractional parameters takes on only rational values, hence the name. Much recent research has been conducted on the properties of these fractional graph parameters (for more information on the uses and applications of fractional graph theory, we refer the reader to [21]).

The following theorem will be important in our analysis.

Theorem 2.2 (*[14]*). *For any vertex-transitive graph G, $\chi_f(G) = \frac{|V(G)|}{\alpha(G)}$.*

Now we define the *circular chromatic number* $\chi_c(G)$.

Definition 2.3 (*[22,27]*). Let k and d be positive integers with $k \geq 2d$. A (k, d)-*colouring* of a graph $G = (V, E)$ on n vertices is a mapping $C : V \to \{0, 1, \dots, k-1\}$ such that $d \leq |C(x) - C(y)| \leq k - d$ for any $xy \in E(G)$. Then, the *circular chromatic number* $\chi_c(G)$ is the infimum of $\frac{k}{d}$ for which there exists a (k, d)-colouring of G.

Note that $\chi(G)$ is just the smallest k for which there exists a $(k, 1)$-colouring of G. So $\chi_c(G)$ is a generalization of $\chi(G)$, where $\chi_c(G) \leq \chi(G)$ for all G. The circular chromatic number is sometimes referred to as the *star chromatic number* [22,27]. An extensive survey of important results and applications of circular chromatic numbers is found in [28].

The following theorems are well known and straightforward to show.

Theorem 2.4 (*[22]*). *For any graph G, $\chi(G) = \lceil \chi_c(G) \rceil$.*