Discrete Mathematics 309 (2009) 2233–2238

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Edge choosability of planar graphs without 5-cycles with a chord[☆]

Yongzhu Chen, Weiyi Zhu, Weifan Wang [*]

Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China

ARTICLE INFO

Article history:
Received 24 September 2007
Received in revised form 28 April 2008
Accepted 28 April 2008
Available online 20 June 2008

Keywords:
Plane graph
Edge choosability
Cycle
Chord

ABSTRACT

Let G be a plane graph having no 5-cycles with a chord. If either $\Delta \geq 6$, or $\Delta = 5$ and G contains no 4-cycles with a chord or no 6-cycles with a chord, then G is edge-$(\Delta + 1)$-choosable, where Δ denotes the maximum degree of G.

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. An *edge-k-coloring* of G is a mapping ϕ from $E(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that $\phi(e) \neq \phi(e')$ for any adjacent edges e and e' of G. The *chromatic index* $\chi'(G)$ is the smaller integer k such that G has an edge-k-coloring. The mapping L is said to be an *edge assignment* for the graph G if it assigns a list $L(e)$ of possible colors to each edge e of G. If G has a proper edge coloring ϕ such that $\phi(e) \in L(e)$ for all edges e, then we say that G is *edge-L-colorable* or ϕ is an *edge-L-coloring* of G. We call G *edge-k-choosable* if it is edge-L-colorable for every edge assignment L satisfying $|L(e)| = k$ for all edges e. The *list chromatic index* $\chi'_l(G)$ of G is the smallest k such that G is edge-k-choosable.

The well-known List Coloring Conjecture is stated as follows:

Conjecture 1. *If G is a multigraph, then $\chi'_l(G) = \chi'(G)$.*

Conjecture 1 is proved for a few special cases, such as bipartite multigraphs [3], complete graphs of odd order [4], multicircuits [15], graphs with $\Delta \geq 12$ and embeddable in a surface of nonnegative characteristic [2], outerplanar graphs [11], etc.

A weaker conjecture on list edge coloring was proposed by Vizing (see [7]):

Conjecture 2. *Every simple graph G is edge-$(\Delta + 1)$-choosable.*

Harris [5] proved that $\chi'_l(G) \leq 2\Delta - 2$ if G is a graph with $\Delta \geq 3$, which implies Conjecture 2 for the case $\Delta = 3$. In his earlier paper [9], Vizing gave a list coloring version of Brooks' theorem, which also confirms Conjecture 2 for a $\Delta = 3$ graph. In 1999, Juvan, Mohar, and Škrekovski [6] settled the case for $\Delta = 4$. Conjecture 2 is also confirmed for complete graphs [4], graphs with girth at least $8\Delta(\ln \Delta + 1.1)$ [7], and planar graphs with $\Delta \geq 9$ [1]. Wang and Lih [12] proved that a planar graph G with $\Delta \geq 6$ and without intersecting 3-cycles is edge-$(\Delta + 1)$-choosable. Suppose that G is a planar graph without k-cycles for some fixed integer $3 \leq k \leq 6$. Then, Conjecture 2 holds if G satisfies one of the following conditions: (i) either $k = 3$, or $k = 4$ and $\Delta \geq 6$ [16]; (ii) $k = 5$ [13]; (iii) $k = 6$ and $\Delta \geq 6$ [10]; (iv) $\Delta = 5$ and either $k = 4$ or $k = 6$ [14].

☆ This research was supported partially by NSFC (No. 10771197) and ZJNSFC (No. Y607467).

* Corresponding author.
E-mail address: wwf@zjnu.cn (W. Wang).

0012-365X/$ – see front matter
doi:10.1016/j.disc.2008.04.056

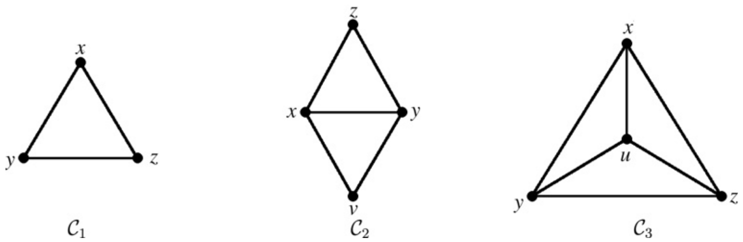

Fig. 1. All possible clusters in G.

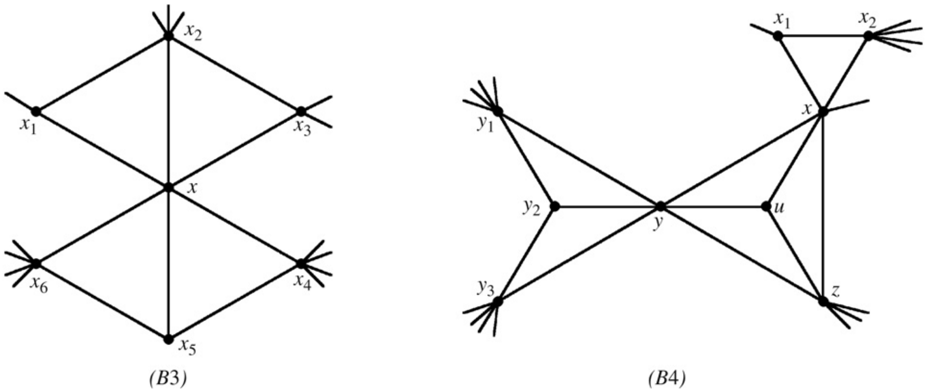

Fig. 2. Configurations (B3) and (B4) in Lemma 5.

A cycle C of length k of a graph G is called a k-*hole* (or k-*net*) if C has no (or at least one) chord in G. In this paper, we shall prove the following results:

Theorem 1. *Every plane graph G with $\Delta \geq 6$ and without 5-nets is edge-$(\Delta + 1)$-choosable.*

Theorem 2. *Every plane graph G with $\Delta = 5$ and without 5- and 6-nets is edge- 6-choosable.*

Theorem 3. *Every plane graph G with $\Delta = 5$ and without 4- and 5-nets is edge- 6-choosable.*

Only simple graphs are considered in this paper. A *plane* graph is a particular drawing of a planar graph in the Euclidean plane. Given a plane graph G, we use $F(G)$ to denote the set of faces of G. For $x \in V(G) \cup F(G)$, let $d(x)$ denote the degree of x in G. A vertex (or face) of degree k is called a k-*vertex* (or k-*face*). For $v \in V(G)$ and $k \geq 3$, let $F_k(v)$ denote the set of k-*faces* in G which are incident to the vertex v. A face $f \in F(G)$ is written as $f = [u_1u_2 \cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are the boundary vertices of f in clockwise order. A 3-face f of G is called an (i, j, k)-*face* if the boundary vertices of f are of degrees i, j, k, respectively. Let $F_3'(v)$ denote the set of all $(3, 6, 6)$-faces in $F_3(v)$. Let $n_3(v)$ denote the number of 3-vertices adjacent to a vertex v.

2. Proof of Theorem 1

Let G be a plane graph with the minimum degree $\delta(G) \geq 3$ and without 5-nets. Then the following configurations H_1 and H_2 will be excluded from G.

(H1) a 3-face adjacent to a 4-face;

(H2) a 3-face adjacent to two nonadjacent 3-faces.

A subgraph $\mathcal{C}$ of G is called a *cluster* if $\mathcal{C}$ consists of a nonempty minimal set of 3-faces in G such that no other 3-face is adjacent to a member of this set. A k-*cluster* is a cluster formed by k 3-faces.

Lemma 4. *Suppose that G is a plane graph with $\delta(G) \geq 3$ and without 5-nets. Then the following configurations exhaust all possible clusters of G (see Fig. 1):*

$\mathcal{C}_1$: *a 3-face;*

$\mathcal{C}_2$: *two adjacent 3-faces;*

$\mathcal{C}_3$: *three mutually adjacent 3-faces.*

Let us observe the cluster $\mathcal{C}_3$ which consists of three 3-faces $f_1 = [xyu]$, $f_2 = [yzu]$ and $f_3 = [zxu]$. Clearly, $d(u) = 3$. If $d(x) = d(y) = d(z) = 6$, we write $\mathcal{C}_3$ as $\mathcal{C}_3^{(6)}$.

Lemma 5. *Let G be a plane graph with $\Delta = 6$ and without 5-nets. Then G contains one of the following configurations, where* (B3) *and* (B4) *are depicted in Fig. 2:*

(B1) *an edge xy with $d(x) + d(y) \leq 8$;*