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Note

Finding Hamiltonian cycles in $\{quasi\text{-}claw, K_{1,5}, K_{1,5}+e\}$-free graphs with bounded Dilworth numbers

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ABSTRACT

Let u and v be two vertices in a graph G. We say vertex u dominates vertex v if $N(v) \subseteq N(u) \cup \{u\}$. If u dominates v or v dominates u, then u and v are comparable. The Dilworth number of a graph G, denoted as $Dil(G)$, is the largest number of pairwise incomparable vertices in the graph G. A graph G is called quasi-claw-free if it satisfies the property: $d(x, y) = 2 \Rightarrow$ there exists $u \in N(x) \cap N(y)$ such that $N[u] \subseteq N[x] \cup N[y]$. A graph is called $\{quasi\text{-}claw, K_{1,5}, K_{1,5}+e\}$-free if it is quasi-claw-free and contains no induced subgraph isomorphic to $K_{1,5}$ or $K_{1,5}+e$, where $K_{1,5}+e$ is a graph obtained by joining a pair of nonadjacent vertices in $K_{1,5}$. It is shown that if G is a k ($k \geq 2$)-connected $\{quasi\text{-}claw, K_{1,5}, K_{1,5}+e\}$-free graph with $Dil(G) \leq 2k-1$, then G is Hamiltonian and a Hamiltonian cycle in G can be found in polynomial time.

1. Introduction

We consider only finite undirected graphs without loops and multiple edges. For terminology, notation and concepts not defined here see [3]. If v is a vertex in a graph, the closed neighborhood $N[v]$ of v is defined as $N(v) \cup \{v\}$. For any two distinct vertices x and y in a graph G, $d(x, y)$ denotes the distance in G from x and y. If $S \subseteq V(G)$, then $N(S)$ denotes the neighbors of S, that is, the set of all vertices in G adjacent to at least one vertex in S.

The definition of the Dilworth number of a graph can be found in [7] (also see [5]). Let u and v be two vertices in a graph G. We say vertex u dominates vertex v if $N(v) \subseteq N[u]$. If u dominates v or v dominates u, then u and v are comparable. The Dilworth number of a graph G, denoted as $Dil(G)$, is the largest number of pairwise incomparable vertices in the graph G. A graph G is called $\{H_1, H_2, \ldots, H_k\}$-free if G contains no induced subgraph isomorphic to any H_i, $1 \leq i \leq k$. If $k = 1$ and $H_1 = K_{1,3}$, then G is called claw-free. The concept of quasi-claw-free graphs was introduced by Ainouche [1]. A graph G is called quasi-claw-free if it satisfies the property: $d(x, y) = 2 \Rightarrow$ there exists $u \in N(x) \cap N(y)$ such that $N[u] \subseteq N[x] \cup N[y]$. Obviously, every claw-free graph is quasi-claw-free. A graph G is called $\{quasi\text{-}claw, H_1, H_2, \ldots, H_k\}$-free if G is quasi-claw-free and contains no induced subgraph isomorphic to any H_i, $1 \leq i \leq k$. Clearly, every claw-free graph is $\{quasi\text{-}claw, K_{1,5}, K_{1,5}+e\}$-free, where $K_{1,5}+e$ is a graph obtained by joining a pair of nonadjacent vertices in $K_{1,5}$.

Bertossi [2] proved that determining whether line graphs have Hamiltonian paths is NP-complete. From Bertossi's result, one can easily see that determining whether line graphs have Hamiltonian cycles is NP-complete and therefore finding Hamiltonian cycles in line graphs is a hard problem. Since every line graph is claw-free and every claw-free graph is $\{quasi\text{-}claw, K_{1,5}, K_{1,5}+e\}$-free, the problem of finding Hamiltonian cycles in $\{quasi\text{-}claw, K_{1,5}, K_{1,5}+e\}$-free graphs is also hard. Hence if we currently want to find polynomial time algorithms for Hamiltonian cycles in $\{quasi\text{-}claw, K_{1,5}, K_{1,5}+e\}$-free graphs, we need extra constraints on this family of graphs. The objective of this paper is to prove the following theorem on finding Hamiltonian cycles in $\{quasi\text{-}claw, K_{1,5}, K_{1,5}+e\}$-free graphs having a constraint on their Dilworth numbers.

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Theorem 1. *Let G be a k ($k \geq 2$)-connected {quasi-claw, $K_{1,5}$, $K_{1,5}+e$}-free graph. If $Dil(G) \leq 2k-1$, then G is Hamiltonian and a Hamiltonian cycle in G can be found in polynomial time.*

Obviously, Theorem 1 has the following corollary.

Corollary 1. *Let G be a k ($k \geq 2$)-connected claw-free graph. If $Dil(G) \leq 2k-1$, then G is Hamiltonian and a Hamiltonian cycle in G can be found in polynomial time.*

We need the following additional notation in the remainder of this paper. If C is a cycle of G, let $\overrightarrow{C}$ denote the cycle C with a given orientation. For $u, v \in C$, let $\overrightarrow{C}[u, v]$ denote the consecutive vertices on C from u to v in the direction specified by $\overrightarrow{C}$. The same vertices, in reverse order, are given by $\overleftarrow{C}[v, u]$. Both $\overrightarrow{C}[u, v]$ and $\overleftarrow{C}[v, u]$ are considered as paths and vertex sets. If u is on C, then the predecessor, successor, next predecessor and next successor of u along the orientation of C are denoted by u^-, u^+, u^{--} and u^{++} respectively. If $A \subseteq V(C)$, then A^- and A^+ are defined as $\{v^- : v \in A\}$ and $\{v^+ : v \in A\}$ respectively. If H is a connected component of a graph G and u and v are two vertices in H, let uHv denote a shortest path between u and v in H which can be found in polynomial time by using the breadth-first search algorithm (see Chapter 22 in [6]).

2. Lemmas

Based on Tarjan's depth-first search algorithm in [10], Köhler proved the following Lemma 1 in [8]. Lemma 1 was used by Brandstädt et al. in [4]. We will also use Lemma 1 in our proofs in Section 3.

Lemma 1. *Let G be a 2-connected graph, and let x, y be two different nonadjacent vertices of G. Then one can construct in linear time (of $|V(G)|$ and $|E(G)|$) two induced, internally disjoint paths, both joining x and y.*

Lemma 2. *Let G be a connected quasi-claw-free graph of order n. Suppose that G has a cycle C of length r, $4 \leq r \leq n-1$. Let u be a vertex on C such that it has a neighbor in $V(G)-V(C)$. Then either $u^-u^+ \in E$ or a cycle of length at least $r+1$ in G can be found in polynomial time.*

Proof of Lemma 2. Let G be a connected quasi-claw-free graph of order n. Suppose that G has a cycle C of length r with a fixed orientation, $4 \leq r \leq n-1$. Let u be a vertex on C such that it has a neighbor in $V(G)-V(C)$. If $u^-u^+ \in E$, then we are done. Now we assume that $u^-u^+ \notin E$. Choose a vertex $x \in N(u) \cap (V(G)-V(C))$. If $xu^- \in E$ or $xu^+ \in E$, we can easily construct a cycle of length $r+1$ in G. Now we assume that $xu^- \notin E$ and $xu^+ \notin E$.

Since $d(x, u^+) = 2$, there exists a vertex v such that $v \in N(x) \cap N(u^+)$ and $N[v] \subseteq N[x] \cup N[u^+]$. Since $u^- \in N[u]$ but $u^- \notin N[x] \cup N[u^+]$, we have $v \neq u$. If $v \in V(C)$, then $v^+ \in N[v] \subseteq N[x] \cup N[u^+]$. Thus $v^+ \in N(x)$ or $v^+ \in N(u^+)$. In either case, we can easily construct a cycle of length $r+1$ in G. Now we assume that $v \in V(G)-V(C)$. In this case we can also easily construct a cycle of length at least $r+1$ in G. $\square$

3. The Proof of Theorem 1

Proof of Theorem 1. Let G be a graph satisfying the conditions in Theorem 1. Check whether G has a pair of nonadjacent vertices. If we cannot find a pair of nonadjacent vertices in G, then G is a complete graph and we can easily find a Hamiltonian cycle in G. If we can find a pair of nonadjacent vertices in G, apply Lemma 1; we can find a cycle C of length $r \geq 4$ in G. This step can be completed in $O(|V(G)|+|V(E)|)$ time. If $r = |V(G)|$, then we are finished. We now assume that $r \leq |V(G)|-1$ and give an orientation on C.

Using a depth-first search algorithm in the graph $G[V(G)-V(C)]$, we can find a connected component, say H, in $G[V(G)-V(C)]$. More details on applying a depth-first search algorithm to find a connected component in a graph can be found in Algorithm 8.3 on page 330 in [9]. This step can be completed in $O(|V(G)|+|V(E)|)$ time. Next we will show that a cycle of length at least $r+1$ in G can be constructed in polynomial time.

Find all the neighbors of $V(H)$ on $V(C)$. We assume that $N(V(H)) \cap V(C) = \{a_1, a_2, \ldots, a_l\}$ such that $h_ia_i \in E$, where $h_i \in V(H)$ for each i, $1 \leq i \leq l$, and $a_1, a_2, \ldots, a_l$ are labeled in the order of the orientation of C. Since G is k ($k \geq 2$)-connected, $l \geq k$. If $a_i^-a_i^+ \notin E$ for some i, $1 \leq i \leq l$, we can apply Lemma 2 to construct a cycle of length at least $r+1$ in G. From now on, we assume that $a_i^-a_i^+ \in E$ for each i, $1 \leq i \leq l$. Clearly, if $2 \leq |\overrightarrow{C}[a_i, a_{i+1}]| \leq 4$ for some i, $1 \leq i \leq l$, where the index $(l+1)$ is regarded as 1, we can easily construct a cycle of length at least $r+1$ in G. From now on, we also assume that $|\overrightarrow{C}[a_i, a_{i+1}]| \geq 5$ for each i, $1 \leq i \leq l$.

Notice first that if $a_ia_{i+1}^- \in E$ for some i, $1 \leq i \leq l$, we can construct a cycle

$$h_ia_i\overleftarrow{C}[a_{i+1}^-, a_i^+]\overleftarrow{C}[a_i^-, a_{i+1}]h_{i+1}Hh_i$$