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Faulhaber's theorem on power sums

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ABSTRACT

We observe that the classical Faulhaber's theorem on sums of odd powers also holds for an arbitrary arithmetic progression, namely, the odd power sums of any arithmetic progression $a+b, a+2b, \ldots, a+nb$ is a polynomial in $na+n(n+1)b/2$. While this assertion can be deduced from the original Fauhalber's theorem, we give an alternative formula in terms of the Bernoulli polynomials. Moreover, by utilizing the central factorial numbers as in the approach of Knuth, we derive formulas for r-fold sums of powers without resorting to the notion of r-reflective functions. We also provide formulas for the r-fold alternating sums of powers in terms of Euler polynomials.

1. Introduction

The classical theorem of Faulhaber states that the sums of odd powers

$$1^{2m-1}+2^{2m-1}+\cdots+n^{2m-1}$$

can be expressed as a polynomial of the triangular number $T_n = n(n+1)/2$; See Beardon [2], Knuth [12]. Moreover, Faulhaber observed that the r-fold summation of n^m is a polynomial in $n(n+r)$ when m is positive and $m-r$ is even [12]. The classical Faulhaber theorem for odd power sums was proved by Jacobi [11]; See also Edwards [4]. Let us recall the notation on the r-fold power sums: $\sum^0 n^m = n^m$, and

$$\sum^r n^m = \sum^{r-1} 1^m + \sum^{r-1} 2^m + \cdots + \sum^{r-1} n^m. \tag{1.1}$$

For example, $\sum^1 n^m = 1^m + 2^m + \cdots + n^m$, and

$$\sum^2 n^m = \sum^1 1^m + \sum^1 2^m + \cdots + \sum^1 n^m = \sum_{i=1}^{n} (n+1-i) i^m.$$

For even powers, it has been shown that the sum $1^{2m}+2^{2m}+\cdots+n^{2m}$ is a polynomial in the triangular number T_n, multiplied by a linear factor in n. Gessel and Viennot [6] had a remarkable discovery that the alternating sum $\sum_{i=1}^{n}(-1)^{n-i} i^{2m}$ is also a polynomial in the triangular number T_n.

Faulhaber's theorem has drawn much attention from various points of view. Grosset and Veselov [7] investigated a generalization of the Faulhaber polynomials related to elliptic curves. Warnaar [15], Schlosser [14] and Zhao and Feng [16]

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studied the q-analogues of the formulas for the first few power sums. Garrett [5] found a combinatorial proof of the formula for sums of q-cubes. Guo and Zeng [8] obtained the q-analogue formula in the general case. Furthermore, Guo, Rubey and Zeng [9] have shown that the q-Faulhaber and q-Salié coefficients are nonnegative and symmetric in a combinatorial setting of nonintersecting lattice paths.

In this paper, we first formulate Faulhaber's theorem in a more general framework, that is, in terms of power sums of an arithmetic progression [10]. Given an arithmetic progression:

$$a+b, a+2b, \ldots, a+nb,$$

Faulhaber's theorem implies that odd power sums of the above series are polynomials in $na+n(n+1)b/2$. In particular, an odd power sum of the first n odd numbers

$$1^{2m-1}+3^{2m-1}+\cdots+(2n-1)^{2m-1}$$

is a polynomial in n^2, and the sum

$$1^{2m-1}+4^{2m-1}+7^{2m-1}+\cdots+(3n-2)^{2m-1}$$

is a polynomial in the pentagonal number $n(3n-1)/2$.

Because of the relation $(a+bi)^m=b^m(a/b+i)^m$, there is no loss of generality to consider the series

$$x+1, x+2, \ldots, x+n. \tag{1.2}$$

Let

$$\lambda=n(n+2x+1) \tag{1.3}$$

be the sum of the sequence $x+1, x+2, \ldots, x+n$. Then the power sums

$$S_{2m-1}=(x+1)^{2m-1}+(x+2)^{2m-1}+\cdots+(x+n)^{2m-1}$$

is a polynomial in λ. For exmaple,

$$S_3=\frac{\lambda^2}{4}+\frac{(x^2+x)}{2}\lambda;$$
$$S_5=\frac{\lambda^3}{6}+\frac{1}{12}(6x^2+6x-1)\lambda^2+\frac{1}{6}(3x^4+6x^3+2x^2-x)\lambda;$$
$$S_7=\frac{\lambda^4}{8}+\frac{1}{6}(3x^2+3x-1)\lambda^3+\frac{1}{12}(9x^4+18x^3+3x^2-6x+1)\lambda^2+\frac{1}{6}(3x^6+9x^5+6x^4-3x^3-2x^2+x)\lambda.$$

It should be noticed that the above more general setting of Faulhaber's theorem can be deduced from the original version of Faulhaber's theorem. When x is a positive integer, we have the relation

$$\sum_{i=1}^{n}(x+i)^m=\sum_{i=1}^{n+x} i^m-\sum_{i=1}^{x} i^m.$$

By Faulhaber's theorem, $\sum_{i=1}^{n+x} i^m$ and $\sum_{i=1}^{x} i^m$ are polynomials in $(n+x)(n+x+1)$ and $x(x+1)$, respectively. Using the following relation

$$(n+x)(n+x+1)=n(n+2x+1)+x(x+1), \tag{1.4}$$

we see that

$$[(n+x)(n+x+1)]^i-[x(x+1)]^i=\sum_{k=1}^{i}\binom{i}{k}[n(n+2x+1)]^k[x(x+1)]^{i-k}, \tag{1.5}$$

which is a polynomial in $n(n+2x+1)$. Clearly, one sees that the above assertion holds for all real numbers x.

Although in principle, Faulhaber's theorem is valid for any arithmetic progression, from a computational point of view it still seems worthwhile to find a formula for the coefficients in terms of Bernoulli polynomials. The main result of this paper is an approach to the study of the r-fold sums of powers without resorting to the properties of r-reflective functions, as in the approach of Knuth [12]. In the last section, we obtain formulas for the r-fold alternating sums of powers in terms of the Euler polynomials.