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Common circulant homogeneous factorisations of the complete digraph☆

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ABSTRACT

In this paper we determine the positive integers n and k for which there exists a homogeneous factorisation of a complete digraph on n vertices with k 'common circulant' factors. This means a partition of the arc set of the complete digraph K_n into k circulant factor digraphs, such that a cyclic group of order n acts regularly on the vertices of each factor digraph whilst preserving the edges, and in addition, an overgroup of this permutes the factor digraphs transitively amongst themselves. This determination generalises a previous result for self-complementary circulants.

1. Introduction

In this paper we determine the positive integers n and k for which there exists a homogeneous factorisation of a complete digraph on n vertices with k 'common circulant' factors, that is, there is a cyclic group of order n that acts regularly on the vertices and preserves the edges of each factor. This determination, in Theorem 2, generalises for homogeneous factorisations a result of Fronček, Rosa, and Širáň in [3] about self-complementary circulants, in much the same way that the result [7, Theorem 1.1] (which motivated our work) generalised a theorem of Muzychuk [11] characterising orders of self-complementary vertex-transitive graphs. In Theorem 3 we draw from these results a characterisation of integers n for which there exists a homogeneous factorisation of a complete digraph on n vertices, and those integers n for which there exists a common circulant such factorisation. We also determine the analogous results for the complete (undirected) graph. In the remainder of this introductory section we introduce the concepts of homogeneous factorisation and common circulant homogeneous factorisation, and state our main results.

1.1. Homogeneous factorisations

A *digraph* or *directed graph*, $\Gamma = (V\Gamma, A\Gamma)$ consists of a set of *vertices* $V\Gamma$, and a set of *arcs* $A\Gamma$, where an arc is an ordered pair of distinct vertices. Thus $A\Gamma \subseteq V\Gamma^{(2)} = \{(\alpha, \beta) \mid \alpha, \beta \in V\Gamma, \alpha \neq \beta\}$. A *factorisation* of a digraph Γ is a partition $\mathcal{P} = \{P_1, \ldots, P_k\}$ of the arc set with at least two parts. This gives rise to *factor digraphs*, $\Gamma_i = (V\Gamma, P_i)$. A *homogeneous factorisation* of a digraph Γ on vertex set Ω, is a factorisation $\mathcal{P}$ such that the following conditions hold.

1. There exist transitive permutation groups M and G with $M < G \leq \mathrm{Aut}(\Gamma) \leq \mathrm{Sym}(\Omega)$.

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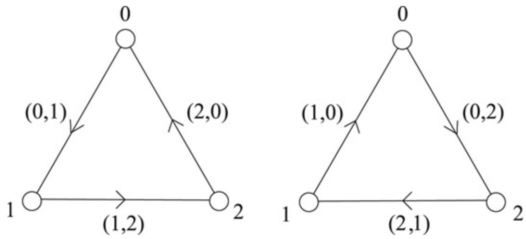

Fig. 1. The factor digraphs of a homogeneous factorisation of degree 3 and index 2.

2. $\mathcal{P}$ is G-invariant (that is, for each $P \in \mathcal{P}$, the image $P^g := \{(\alpha^g, \beta^g) : (\alpha, \beta) \in P\}$ is also a part of $\mathcal{P}$), and the induced action of G on $\mathcal{P}$ is transitive.
3. The group M fixes setwise each part of $\mathcal{P}$ (or equivalently, the induced action of M on $\mathcal{P}$ is trivial).

Since M fixes P_i setwise we have $M \leq \text{Aut}(\Gamma_i)$ for each i, and thus the factor digraphs Γ_i are M-vertex transitive. Also because G acts transitively on $\mathcal{P}$, the factor digraphs are pairwise isomorphic. A homogeneous factorisation can be denoted by a quadruple, $(M, G, \Gamma, \mathcal{P})$. The *complete digraph*, K_n, is a digraph with n vertices and $A\Gamma = V\Gamma^{(2)}$. In this paper we consider factorisations of the complete digraph, and we denote the homogeneous factorisation by $(M, G, \Omega, \mathcal{P})$, where Ω is the vertex set of the complete digraph under consideration.

The *degree* of a homogeneous factorisation $(M, G, \Gamma, \mathcal{P})$ is the number of vertices of the digraph Γ. The number of parts in $\mathcal{P}$ is called the *index* of the factorisation. In particular, in a homogeneous factorisation of a complete digraph of degree n and index 2, the factor digraphs are vertex-transitive self-complementary digraphs on n vertices. A homogeneous factorisation of a complete digraph of degree n and index k is a generalisation of this. We illustrate this concept, as we will do for subsequent concepts, with the smallest example.

Example 1. See Fig. 1. Let $\Omega = \{0, 1, 2\}$, $P_1 = \{(0, 1), (1, 2), (2, 0)\}$, $P_2 = \{(0, 2), (2, 1), (1, 0)\}$, $M = \langle(012)\rangle$, $G = \langle M, (12)\rangle$, and $\mathcal{P} = \{P_1, P_2\}$. Then $(M, G, \Omega, \mathcal{P})$ is a homogeneous factorisation of degree 3 and index 2.

A factorisation is *symmetric* if for each factor P_i, we have $(\alpha, \beta) \in P_i$ if and only if $(\beta, \alpha) \in P_i$. A digraph Γ is *undirected*, or simply a *graph* when $(\alpha, \beta) \in A\Gamma$ if and only if $(\beta, \alpha) \in A\Gamma$. Then the arcs (α, β) and (β, α) can be considered as an unordered pair $\{\alpha, \beta\}$, called an *edge*. Thus in a symmetric factorisation, the factor digraphs are considered undirected and this is equivalent to a factorisation of the edge set of a graph.

Remark 1. A homogeneous factorisation of the complete digraph corresponds to a transitive orbital decomposition or k-TOD, as described by Li and Praeger in [7]. They give many results about k-TODs which can be translated into the language of homogeneous factorisations. One key result is the following, which is used in our proofs below. If there exists a homogeneous factorisation of degree n and index k, then by [7, Lemma 2.5] we have that $n \equiv 1 \pmod{k}$. If there exists a symmetric homogeneous factorisation of degree n and index k, then by [7, Lemma 2.5] we have that $n \equiv 1 \pmod{2k}$. Note that Fig. 1 is not symmetric, and the degree and index satisfy the first condition, that is $3 \equiv 1 \pmod{2}$, but by the second condition there is no symmetric homogeneous factorisation of degree 3.

1.2. Cyclic homogeneous factorisations

If $(M, G, \Omega, \mathcal{P})$ is a symmetric homogeneous factorisation of degree n and index 2, with $\mathcal{P} = \{P_1, P_2\}$, then $G^{\mathcal{P}} \cong \mathbb{Z}_2$ and $\Gamma_i := (\Omega, P_i)$, for $i = 1, 2$, are a pair of vertex-transitive, self-complementary (undirected) graphs on n vertices. In 1998 Muzychuk [11] proved that such a factorisation exists if and only if the following condition Hom$(n, 4)$ holds, where n_r denotes the r-part of n, for a prime r dividing n, namely the highest power of r dividing n.

$$\text{Hom}(n, 4): \quad \forall \text{ primes } r \text{ dividing } n,\ n_r \equiv 1 \pmod{4}.$$

To prove his result, Muzychuk devised a technique of reducing the self-complementary graphs Γ_i to a set of so-called Sylow subgraphs, which were also vertex-transitive and self-complementary, and had a prime power number of vertices. In 2003 Li and Praeger extended this result for *cyclic homogeneous factorisations* of arbitrary index k, that is, for homogeneous factorisations $(M, G, \Omega, \mathcal{P})$ of index k such that the induced group $G^{\mathcal{P}} \cong \mathbb{Z}_k$. Their result involved the following extension of Muzychuk's condition, namely, for a positive integer k,

$$\text{Hom}(n, k): \quad \forall \text{ primes } r \text{ dividing } n,\ n_r \equiv 1 \pmod{k}.$$

The following theorem is their result stated in the language of homogeneous factorisations.

Theorem 1 *([7, Theorem 1.1]). Let n and k be integers such that $n \geq 3$ and $k \geq 2$.*

1. *There exists a cyclic homogeneous factorisation of degree n and index k if and only if* Hom(n, k) *holds.*