Discrete Mathematics 309 (2009) 3255–3269

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Graphs, designs and codes related to the n-cube

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ARTICLE INFO

Article history:
Received 17 March 2008
Received in revised form 22 August 2008
Accepted 15 September 2008
Available online 4 October 2008

Keywords:
Graph
Design
Code
Permutation decoding

ABSTRACT

For integers $n \geq 1, k \geq 0$, and $k \leq n$, the graph Γ_n^k has vertices the 2^n vectors of $\mathbb{F}_2^n$ and adjacency defined by two vectors being adjacent if they differ in k coordinate positions. In particular Γ_n^1 is the n-cube, usually denoted by Q_n. We examine the binary codes obtained from the adjacency matrices of these graphs when $k = 1, 2, 3$, following the results obtained for the binary codes of the n-cube in Fish [Washiela Fish, Codes from uniform subset graphs and cyclic products, Ph.D. Thesis, University of the Western Cape, 2007] and Key and Seneviratne [J.D. Key, P. Seneviratne, Permutation decoding for binary self-dual codes from the graph Q_n where n is even, in: T. Shaska, W. C Huffman, D. Joyner, V. Ustimenko (Eds.), Advances in Coding Theory and Cryptology, in: Series on Coding Theory and Cryptology, vol. 2, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007, pp. 152–159]. We find the automorphism groups of the graphs and of their associated neighbourhood designs for $k = 1, 2, 3$, and the dimensions of the ternary codes for $k = 1, 2$. We also obtain 3-PD-sets for the self-dual binary codes from Γ_n^2 when $n \equiv 0 \pmod 4$, $n \geq 8$.

1. Introduction

In Fish [6] and Key and Seneviratne [12], the binary codes obtained from the row span over $\mathbb{F}_2$ of an adjacency matrix for the n-cube Q_n were examined, and the codes in the case of n even found to be self-dual with minimum weight n. Further, 3-PD-sets were found in [12] for partial permutation decoding. The n-cubes belong to the class of graphs Γ_n^k, for $n \geq 1$, $k \geq 0$ integers and $k \leq n$, with vertices the 2^n vectors of $\mathbb{F}_2^n$ and adjacency defined by two vectors being adjacent if they differ in k coordinate positions. The n-cube is Γ_n^1, which is also a Hamming graph, $H(n, 2)$.

In this paper we will examine the binary codes from an adjacency matrix for the graphs Γ_n^k for $k = 2, 3$. We show that for $n \equiv 0 \pmod 4$ the codes from Γ_n^2 are self-dual and, when the same point ordering is used, distinct from those from the n-cube $\Gamma_n^1 = Q_n$: see Proposition 1, Lemma 3 and Proposition 8. We obtain the dimensions of these codes, and also those of the ternary codes for Γ_n^1 and Γ_n^2: see Propositions 6 and 7. The automorphism groups of the codes (see Section 2 for our terminology) contain those of the defining graph and design; we identify the groups of the graphs and designs in Propositions 3 and 4.

We summarize in a theorem what we have found for the dimensions of the binary codes for $k = 1, 2, 3$, including the result for the binary codes for $k = 1$ for completeness (see Result 2). We also include our results on the ternary codes for $k = 1, 2$, noting that the ternary codes for $k = 3$ seem to be quite different and to merit separate study. We include our results on the automorphism groups of the graphs and designs. In the theorem we have used the same point ordering for the vectors of $\mathbb{F}_2^n$ for the graphs Γ_n^k for distinct k in order to compare the codes.

Theorem 1. *For integers $n \geq 1, k \geq 0$, and $n \geq k$, let Γ_n^k denote the graph with vertices the 2^n vectors of $\mathbb{F}_2^n$ and adjacency defined by two vectors being adjacent if they differ in k coordinate positions. Let $C_p(\Gamma_n^k)$ denote the p-ary code obtained by the*

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doi:10.1016/j.disc.2008.09.024

row span of an adjacency matrix for Γ_n^k over $\mathbb{F}_p$ where p is a prime. Let $\mathcal{D}_n^k$ denote the 1-design with points the vertices of Γ_n^k and blocks given by the set of neighbours of each vertex.

1. *For $p = 2$:*
   (a) *$C_2(\Gamma_n^1)$ has dimension 2^n for n odd, and dimension 2^{n-1} for n even. Further, the code is self-dual and has minimum weight n if n is even.*
   (b)
$$\dim(C_2(\Gamma_n^2)) = \begin{cases} 2^{n-1} & \text{for } n \equiv 0 \pmod 4 \\ 2^n & \text{for } n \equiv 2, 3 \pmod 4 \\ 2^{n-1} - 2^{\frac{n-1}{2}} & \text{for } n \equiv 1 \pmod 4. \end{cases}$$
   Furthermore, $C_2(\Gamma_n^2)$ is self-dual for $n \equiv 0 \pmod 4$, self-orthogonal for $n \equiv 1 \pmod 4$. For $n \equiv 0 \pmod 4$, $n \geq 8$, $\dim(C_2(\Gamma_n^1) \cap C_2(\Gamma_n^2)) = 2^{n-2} + 2^{\frac{n}{2}-1}$.
   (c) *For $n \geq 2$,*
$$\dim(C_2(\Gamma_n^3)) = \begin{cases} 2^{n-1} & \text{for } n \equiv 0 \pmod 4, C_2(\Gamma_n^3) = C_2(\Gamma_n^1) \\ 2^{n-1} - 2^{\frac{n-1}{2}} & \text{for } n \equiv 1 \pmod 4, C_2(\Gamma_n^3) = C_2(\Gamma_n^2) \\ 2^{n-2} - 2^{\frac{n-2}{2}} & \text{for } n \equiv 2 \pmod 4, C_2(\Gamma_n^3) \subset C_2(\Gamma_n^1) \\ 2^n & \text{for } n \equiv 3 \pmod 4. \end{cases}$$
2. *For $p = 3$:*
   (a)
$$\dim(C_3(\Gamma_n^1)) = \begin{cases} \frac{2}{3}(2^n - 1) & \text{if } n \text{ is even} \\ \frac{2}{3}(2^n + 1) & \text{if } n \text{ is odd} \end{cases}$$
   (b)
$$C_3(\Gamma_n^2) = \begin{cases} C_3(\Gamma_n^1) & \text{for } n \equiv 0 \pmod 3 \\ C_3(\Gamma_n^1)^\perp & \text{for } n \equiv 1 \pmod 3 \\ \mathbb{F}_3^{2^n} & \text{for } n \equiv 2 \pmod 3. \end{cases}$$
   Furthermore $C \cap C^\perp = \{0\}$ for C any of these ternary codes.
3. *If T denotes the translation group on the vector space $\mathbb{F}_2^n$, T^* the subgroup of T of translations of even weight vectors, and S_n is the symmetric group of degree n, then $\mathrm{Aut}(\Gamma_n^1) = T \rtimes S_n$, and, for $n \geq 6$,*
$$\mathrm{Aut}(\mathcal{D}_n^1) = \mathrm{Aut}(\mathcal{D}_n^2) = \mathrm{Aut}(\Gamma_n^2) = (T^* \rtimes S_n) \wr S_2,$$
   and for $n \geq 8$,
$$\mathrm{Aut}(\mathcal{D}_n^3) = \mathrm{Aut}(\mathcal{D}_n^1), \qquad \mathrm{Aut}(\Gamma_n^3) = \mathrm{Aut}(\Gamma_n^1).$$

The proof of the theorem follows from the propositions in the following sections. In addition, as in [6,12], we obtain 2- and 3-PD-sets for the self-dual binary codes from Γ_n^2 in Proposition 5.

Sections 2 and 3 give the necessary background material and definitions. Sections 4 and 5 give the results for the binary codes of Γ_n^k for $k = 1, 2$. Section 6 finds the automorphism groups of the designs and graphs. In Section 7 we find 3-PD-sets for the self-dual binary code of Γ_n^2 when $n \equiv 0 \pmod 4$. Sections 8 and 9 deal with the ternary codes for Γ_n^k for $k = 1, 2$, and the final sections look at the dual codes in the binary and ternary cases.

2. Background and terminology

The notation for designs and codes is as in [1]. An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{J})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{J}$ is a t-(v, k, λ) design, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points, and every t distinct points are together incident with precisely λ blocks. The design is **symmetric** if it has the same number of points and blocks. The **code $C_F(\mathcal{D})$ of the design** $\mathcal{D}$ over the finite field F is the space spanned by the incidence vectors of the blocks over F. If $\mathcal{Q}$ is any subset of $\mathcal{P}$, then we will denote the **incidence vector** of $\mathcal{Q}$ by $\mathbf{v}^{\mathcal{Q}}$. If $\mathcal{Q} = \{P\}$ where $P \in \mathcal{P}$, then we will write v^P instead of $v^{\{P\}}$. Thus $C_F(\mathcal{D}) = \langle v^B \mid B \in \mathcal{B} \rangle$, and is a subspace of $F^{\mathcal{P}}$, the full vector space of functions from $\mathcal{P}$ to F. If $F = \mathbb{F}_p$ then the p-**rank** of the design, written $\mathrm{rank}_p(\mathcal{D})$, is the dimension of its code $C_F(\mathcal{D})$, which we usually write as $C_p(\mathcal{D})$.

All the codes here are **linear codes**, and the notation $[n, k, d]_q$ will be used for a q-ary code C of length n, dimension k, and minimum weight d, where the **weight wt($\boldsymbol{v}$)** of a vector v is the number of non-zero coordinate entries. The **distance d($\boldsymbol{u}, \boldsymbol{v}$)** between two vectors u, v is the number of coordinate positions in which they differ, i.e., $\mathrm{wt}(u - v)$. If $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$, then we write $\boldsymbol{u} \cap \boldsymbol{v} = (\boldsymbol{u}_1\boldsymbol{v}_1, \ldots, \boldsymbol{u}_n\boldsymbol{v}_n)$. A **generator matrix** for C is a $k \times n$ matrix made up of a basis for C, and the **dual** code $C^\perp$ is the orthogonal under the standard inner product $(,)$, i.e. $C^\perp = \{v \in F^n | (v, c) = 0 \text{ for all } c \in C\}$. A code C is **self-dual** if $C = C^\perp$ and, if C is binary, **doubly-even** if all codewords have weight divisible by 4. A **check matrix** for C is a generator matrix for $C^\perp$. The **all-one vector** will be denoted by $\boldsymbol{\jmath}$, and is the vector with all entries equal to 1. Two linear codes of the same length and over the same field are **isomorphic** if they can be obtained from one another by