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Sublattices of product spaces: Hulls, representations and counting

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Abstract

The Cartesian product of lattices is a lattice, called a *product space*, with componentwise meet and join operations. A *sublattice* of a lattice L is a subset closed for the join and meet operations of L. The *sublattice hull* $\mathscr{L}Q$ of a subset Q of a lattice is the smallest sublattice containing Q. We consider two types of representations of sublattices and sublattice hulls in product spaces: *representation by projections* and *representation with proper boundary epigraphs*. We give sufficient conditions, on the dimension of the product space and/or on the sublattice hull of a subset Q, for $\mathscr{L}Q$ to be entirely defined by the sublattice hulls of the two-dimensional projections of Q. This extends results of Topkis (1978) and of Veinott [Representation of general and polyhedral subsemilattices and sublattices of product spaces, Linear Algebra Appl. 114/115 (1989) 681–704]. We give similar sufficient conditions for the sublattice hull $\mathscr{L}Q$ to be representable using the epigraphs of certain *isotone* (i.e., nondecreasing) functions defined on the one-dimensional projections of Q. This also extends results of Topkis and Veinott. Using this representation we show that $\mathscr{L}Q$ is convex when Q is a convex subset in a vector lattice (Riesz space), and is a polyhedron when Q is a polyhedron in $\mathbb{R}^n$.

We consider in greater detail the case of a finite product of finite *chains* (i.e., totally ordered sets). We use the representation with proper boundary epigraphs and provide upper and lower bounds on the number of sublattices, giving a partial answer to a problem posed by Birkhoff in 1937. These bounds are close to each other in a logarithmic sense. We define a *corner representation* of isotone functions and use it in conjunction with the representation with proper boundary epigraphs to define an encoding of sublattices. We show that this encoding is optimal (up to a constant factor) in terms of memory space. We also consider the *sublattice hull membership problem* of deciding whether a given point is in the sublattice hull $\mathscr{L}Q$ of a given subset Q. We present a *good characterization* and a polynomial time algorithm for this sublattice hull membership problem. We construct in polynomial time a data structure for the representation with proper boundary epigraphs, such that sublattice hull membership queries may be answered in time logarithmic in the size $|Q|$ of the given subset.
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1. Introduction

Lattices and sublattices are fundamental algebraic structures with applications ranging from Economics [12,17] to Optimization [7,8], Graph Theory [6], Engineering [13,14] and other fields (see, e.g., [3–5]). Recall that a *lattice* is a partially ordered set L such that each pair of elements $u, v \in L$ has a greatest lower bound, or *meet*, $u \wedge v \in L$ and a smallest upper bound, or *join*, $u \vee v \in L$. A *sublattice* S of a lattice L is a subset of L closed for the join and meet operations of L.

For many applications, it is often important to be able to represent a (sub)lattice in a computationally or algebraically convenient way. It is also useful to be able to recognize if a given subset Q of a lattice is a sublattice and, if not, to construct its *sublattice hull* $\mathscr{L}Q$, that is, the smallest sublattice containing Q.

Topkis and Veinott [16–18] present several results concerning the representation and recognition of sublattices of *product spaces*, i.e., Cartesian products of lattices with componentwise meet and join operations. Typical examples of product spaces include the Euclidian vector space $\mathbb{R}^n$, the integer lattice $\mathbb{Z}^n$, the Boolean lattice $\mathbb{B}^n = \{0, 1\}^n$ and, more generally, any function space Y^X where Y is a lattice.

The importance of product spaces was demonstrated, among others, by Birkhoff [1,2] who shows that any finite distributive lattice is isomorphic to a sublattice of some Boolean lattice $\mathbb{B}^n$.

Topkis proves that every sublattice L of a *finite* product of lattices can be represented as the intersection of the "cylinders" based on all the two-dimensional projections of L onto coordinate planes. Veinott extends this result to a class of infinite-dimensional product spaces. In Section 2 we present a similar and more general *representation by projections* for sublattice hulls and for a broader class of product spaces.

Topkis also proves that every sublattice L of a finite product space can be represented as the set of all points that are in the Cartesian product $\bigotimes_{i=1}^{n} \pi_i L$ of its projections on the coordinate axes, and that satisfy a certain system of nonlinear inequalities involving at most two variables (coordinates). In Section 3 we refine and extend this result by showing that the sublattice hull of every subset in a broad class of product spaces is the intersection of the cylinders based on the epigraphs of certain single-variable isotone (i.e., nondecreasing) functions, the *boundary functions*. We use this *representation with proper boundary epigraphs* and show that the sublattice hull of a convex (resp., polyhedral) subset is convex (resp., polyhedral).

In 1937 Birkhoff [1] posed the problem of determining the number of sublattices of the Boolean lattice $\mathbb{B}^d$. In Section 4, using the representation by proper boundary epigraphs, we determine upper and lower bounds on the number of sublattices in a finite product of finite chains. (A *chain* is a totally ordered set). These bounds are close to (i.e., within a constant factor of) each other in a logarithmic sense.

In Section 5 we present a *corner representation* of isotone functions and of their epigraphs when the space is a finite product of finite chains. This corner representation of the boundary epigraphs provides us with a way of encoding an arbitrary sublattice of a given product space. Using the base-2 logarithm of the number of sublattices, we show that this sublattice encoding is optimal (up to a constant factor) in terms of memory space required.

In Section 6 we consider the *sublattice hull membership problem* of deciding whether a given point is in the sublattice hull of a given subset of a product space. When the space is a finite product of finite chains, we present a *good characterization* and a polynomial-time algorithm for this sublattice hull membership problem. We also show how to construct in polynomial time a data structure implementing the representation of Section 3 for the sublattice hull of a given subset. This data structure then allows us to answer sublattice hull membership queries in time logarithmic in the subset size.

2. Representation of sublattice hulls by projections

Let I be an arbitrary index set. For all $i \in I$, let T_i be a lattice with join and meet operations denoted by $\vee$ and $\wedge$, respectively, and with associated partial order $\leqslant$ (defined by $u \leqslant v$ iff $u \wedge v = u$). The Cartesian product, or *product space*, $T_I = \bigotimes_{i \in I} T_i$ is the set of all vectors (or points) $x = (x_i)_{i \in I}$ with components $x_i \in T_i$ for all $i \in I$. The product space T_I is a lattice with respect to the operations $\vee$ and $\wedge$ defined componentwise, i.e., $x \vee y = (x_i \vee y_i)_{i \in I}$ and $x \wedge y = (x_i \wedge y_i)_{i \in I}$ for any $x, y \in T_I$. Its associated partial order is then defined by $x \leqslant y$ iff $x_i \leqslant y_i$ for all $i \in I$. A subset L of T_I is called a *sublattice* of T_I if $x \wedge y \in L$ and $x \vee y \in L$ for all $x, y \in L$. The intersection of any family of sublattices of L is a sublattice of L. If Q is an arbitrary subset of T_I, we let $\mathscr{L}Q$ denote the *sublattice hull* of Q, that is, the intersection of all sublattices of T_I that contain Q; thus $\mathscr{L}Q$ is the smallest sublattice of L containing Q. In this

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