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Reconstructing compositions☆

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Abstract

We consider the problem of reconstructing compositions of an integer from their subcompositions, which was raised by Raykova (albeit disguised as a question about layered permutations). We show that every composition w of $n \geqslant 3k+1$ can be reconstructed from its set of k-deletions, i.e., the set of all compositions of $n-k$ contained in w. As there are compositions of $3k$ with the same set of k-deletions, this result is best possible.

MSC: 68R15; 06A07; 05A17; 05C60

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1. Introduction

The Reconstruction Conjecture states that given the multiset of isomorphism types of 1-vertex deletions (briefly, 1-*deletions*) of a graph G—the *deck* of G—on three or more vertices, it is possible to determine G up to isomorphism. The stronger set version of the conjecture due to Harary [5] only allows access to the *set* of 1-deletions and requires G to have four or more vertices. These conjectures can be made even more difficult by considering k-deletions instead of 1-deletions, for which we refer to Manvel [7].

Such reconstruction questions extend naturally to other combinatorial contexts. For example, Schützenberger and Simon (see [6, Theorem 6.2.16]) proved that every word of length $n \geqslant 2k+1$ can be reconstructed from its set of k-deletions (i.e., subwords of length $n-k$). This bound is tight because the words $(ab)^k$ (the word with ab repeated k times) and $(ba)^k$ have the same set of k-deletions: all words of length k over the alphabet $\{a, b\}$. Answering a question of Cameron [4], Pretzel and Siemons [8] considered the partition context, where they proved that every partition of $n \geqslant 2(k+3)(k+1)$ can be reconstructed from its set of k-deletions.

Motivated by a question of Raykova [9] (described at the end of the paper), we consider the problem of set reconstruction for compositions (ordered partitions), establishing the following result.

Theorem 1. *All compositions of* $n \geqslant 3k+1$ *can be reconstructed from their sets of* k*-deletions*.

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Our proof of Theorem 1 illustrates an algorithm to perform the reconstruction. Perhaps more convincing than the proof is the Maple implementation of this algorithm, available from the author's homepage.

2. Notation

We view a composition as a word w whose letters are positive integers, i.e., a word in $\mathbb{P}^*$. We denote the length of w by $|w|$ and the sum of the entries of w by $\|w\|$, and say that w is a composition of $\|w\|$. A 1-deletion of w is a composition that can be obtained either by lowering a $a \geqslant 2$ entry of w by 1 or by removing an entry of w that is equal to 1. A 2-deletion is then a 1-deletion of a 1-deletion, and so on.

This notion naturally defines a partial order[1] on compositions: $u \leqslant w$ if w contains a subword $w(i_1)w(i_2)\cdots w(i_\ell)$ of length $\ell = |u|$ such that $u(j) \leqslant w(i_j)$ for all $1 \leqslant j \leqslant \ell$. (We refer to the indices $i_1 < \cdots < i_\ell$ as an *embedding* of u.) For example, $1211 \leqslant 21\,312$ because of the subword 2312. If $u \leqslant w$ then u is a $(\|w\| - \|u\|)$—deletion of w. Returning to the previous example, $\|21\,312\| = 9$ and $\|1211\| = 5$, so 1211 is a 4-deletion of 21 312.

3. A lower bound

In the context of words, the fact that the sets of k-deletions of $(ab)^k$ and $(ba)^k$ are both equal to the set of all words of length k over $\{a, b\}$ provides a lower bound on k-reconstructibility. Here we can use a very similar example: the sets of k-deletions of $(12)^k$ and $(21)^k$ are both equal to the set of all compositions of $2k$ in which no entry is greater than 2. This implies that Theorem 1 is best possible.

Proof of Theorem 1. Our reconstruction algorithm/proof of Theorem 1 employs several composition statistics. One is the *exceedance number*, defined by $\mathrm{ex}(w) = \|w\| - |w| = \sum (w(i) - 1)$ where the sum is over all entries $w(i)$. Another is the number of ones in w, which can be approximated from its set of k-deletions:

Lemma 2. *The composition w of $n \geqslant 3k + 1$ has at least k ones if and only if either*

(1) *1^{n-k} is a k-deletion of w, or*
(2) *the longest k-deletion of w is k letters longer than the shortest k-deletion of w.*

Moreover, w has precisely k ones if and only if (2) holds and w has a k-deletion without ones.

Proof. It is easy to see that if either (1) or (2) occurs then w has at least k ones. Suppose then that w has at least k ones. If $\mathrm{ex}(w) \leqslant k$ then since $1^{|w|}$ is an $\mathrm{ex}(w)$-deletion of w, it follows that 1^{n-k} is a k-deletion of w, satisfying (1). On the other hand, if $\mathrm{ex}(w) > k$ then some k-deletion of w has length $|w|$, while the fact that w contains at least k ones guarantees that some k-deletion of w has length $|w| - k$, satisfying (2). The second claim in the lemma is then readily verified. □

Given a set of k-deletions of a composition, the first step in our algorithm is to apply Lemma 2 to decide if the composition has fewer than k, precisely k, or more than k ones. The three cases are handled separately. The first two are relatively straightforward, while the last is more delicate.

Lemma 3. *If w is a composition of $n \geqslant 3k + 1$ with fewer than k ones, then w can be reconstructed from its set of k-deletions.*

Proof. Given the set of k-deletions of a composition w satisfying these hypotheses, our algorithm can apply the result of Lemma 2 to determine that w has fewer than k ones. It then follows that

$$\mathrm{ex}(w) \geqslant \frac{\|w\| - (\#\text{ of ones in } w)}{2} \geqslant \frac{2k+2}{2} = k + 1.$$

[1] This partial order was first considered by Bergeron et al. [1], and has since been studied by Snellman [12,13], Sagan and Vatter [10], and Björner and Sagan [2].