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P_5-factorization of complete bipartite graphs[☆]

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Abstract

A P_k-factor of complete bipartite graph $K_{m,n}$ is a spanning subgraph of $K_{m,n}$ such that every component is a path of length k. A P_k-factorization of $K_{m,n}$ is a set of edge-disjoint P_k-factors of $K_{m,n}$ which is a partition of the set of edges of $K_{m,n}$. When k is an even number, the spectrum problem for a P_k-factorization of $K_{m,n}$ has been completely solved. When k is an odd number, Ushio in 1993 proposed a conjecture. However, up to now we only know that Ushio Conjecture is true for $k=3$. In this paper we will show that Ushio Conjecture is true when $k=5$. That is, we shall prove that a necessary and sufficient condition for the existence of a P_5-factorization of $K_{m,n}$ is (1) $3n \geqslant 2m$, (2) $3m \geqslant 2n$, (3) $m+n \equiv 0 \pmod 5$, and (4) $5mn/[4(m+n)]$ is an integer.
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1. Introduction

Let P_k be the path on k vertices and $K_{m,n}$ be the complete bipartite graph with partite sets V_1 and V_2, where $|V_1|=m$ and $|V_2|=n$. A subgraph F of $K_{m,n}$ is called a spanning subgraph of $K_{m,n}$ if F contains all the vertices of $K_{m,n}$. A P_k-factor of $K_{m,n}$ is a spanning subgraph F of $K_{m,n}$ such that every component of F is a P_k and every pair of P_k's has no vertex in common. A P_k-factorization of $K_{m,n}$ is a set of edge-disjoint P_k-factors of $K_{m,n}$ which is a partition of the set of edges of $K_{m,n}$. In paper [6], the P_k-factorization of $K_{m,n}$ is defined as a resolvable $(m, n, k, 1)$ bipartite P_k-design. The graph $K_{m,n}$ is called P_k-factorizable whenever it has a P_k-factorization. For graph theoretical terms, see [4].

When k is an even number, the spectrum problem for a P_k-factorization of $K_{m,n}$ has been completely solved (see [3,6,8]). When k is an odd number, the spectrum problem for a P_k-factorization of $K_{m,n}$ seems to be much less tractable. Ushio in [5] gave a necessary and sufficient condition for existence of P_3-factorization of $K_{m,n}$. Some further work was done by Ushio and Tsuruno in [7], Du in [1,2], and Wang and Du in [9]. In paper [6], Ushio proposed the following conjecture [6, Conjecture 5.3].

Conjecture 1.1. Let m and n be positive integers and k be odd. Then $K_{m,n}$ has a P_k-factorization if and only if (1) $(k+1)n \geqslant (k-1)m$, (2) $(k+1)m \geqslant (k-1)n$, (3) $m+n \equiv 0 \pmod k$, and (4) $kmn/[(k-1)(m+n)]$ is an integer.

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However, up to now we only know that Ushio Conjecture is true for $k = 3$. In this paper we will show that Ushio Conjecture is true when $k = 5$. That is, we shall prove:

Theorem 1.2. *Let m and n be positive integers. Then $K_{m,n}$ has a P_5-factorization if and only if* (1) $3n \geqslant 2m$, (2) $3m \geqslant 2n$, (3) $m + n \equiv 0 \pmod 5$, *and* (4) $5mn/[4(m + n)]$ *is an integer.*

2. Proof of the main result

First, assume that a P_5-factorization of $K_{m,n}$ is given. Certain integers are defined as follows:
t = the number of copies of P_5 in any factor,
r = the number of P_5-factors in the factorization,
a = the number of copies of P_5 with its endpoints in Y in a particular P_5-factor (type M),
b = the number of copies of P_5 with its endpoints in X in a particular P_5-factor(type W),
c = the total number of copies of P_5 in the whole factorization.

Since any P_5-factor spans $K_{m,n}$,

$$t = \frac{m + n}{5}. \tag{2.1}$$

Every P_5-factor has $4t$ edges so that in a factorization $mn = 4rt = 4c$. Thus

$$r = \frac{5mn}{4(m + n)}. \tag{2.2}$$

By definition of a and b, we get $2a + 3b = m$ and $3a + 2b = n$. Hence

$$a = \frac{3n - 2m}{5}, \tag{2.3}$$

$$b = \frac{3m - 2n}{5}. \tag{2.4}$$

Since expressions (2.1)–(2.4) must be integers, we have the following necessary condition for the existence of a P_5-factorization of the complete bipartite graph $K_{m,n}$.

Lemma 2.1. *If $K_{m,n}$ has a P_5-factorization, then* (1) $3n \geqslant 2m$, (2) $3m \geqslant 2n$, (3) $m + n \equiv 0 \pmod 5$, *and* (4) $5mn/[4(m + n)]$ *is an integer.*

The remainder of this section is devoted to the proof of sufficiency theorem 1.2. For any two integers x and y, we use $gcd(x, y)$ to denote the greatest common divisor of x and y. The following lemma is obvious.

Lemma 2.2. *Let g, p and q be positive integers, if $gcd(p, q) = 1$, then*

$$gcd(pq, p + gq) = gcd(p, g).$$

We first prove the following result, which is used later in this paper.

Lemma 2.3. *If $K_{m,n}$ has a P_5-factorization, then $K_{sm,sn}$ has a P_5-factorization for every positive integer s.*

Proof. Let $\{F_i : 1 \leqslant i \leqslant s\}$ be a P_2-factorization of $K_{s,s}$ (whose existence, see [4]). For each $i \in \{1, 2, \ldots, s\}$, replace every edge of F_i by a $K_{m,n}$ to get a factor G_i of $K_{sm,sn}$ such that the graph G_i are pairwise edge-disjoint and their union is $K_{sm,sn}$. Since $K_{m,n}$ has a P_5-factorization, it is clear that the graph G_i, too, has a P_5-factorization. Consequently, $K_{sm,sn}$ has a P_5-factorization. This proves the theorem. □

Now we start to prove our main result Theorem 1.2. There are three cases to consider.

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