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Note

Asymmetric marking games on line graphs[☆]

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Abstract

This paper investigates the asymmetric marking games on line graphs. Suppose G is a graph with maximum degree Δ and G has an orientation with maximum outdegree k, we show that the $(a, 1)$-game coloring number of the line graph of G is at most $\Delta + 2k + \lceil \frac{k}{a} \rceil - 1$. When $a = 1$, this improves some known results of the game coloring number of the line graphs.

MSC: 05C15

Keywords: Game coloring number; Line graph; Harmonious strategy

1. Introduction

For a graph $G=(V, E)$, the neighborhood of a vertex v is denoted by $N_G(v)$. Let $O(G)$ be the set of all orientations of G. For an orientation $\vec{G}$ of G and for a vertex v of $\vec{G}$, we denote the outneighborhood of v in $\vec{G}$ by $N^+_{\vec{G}}(v) = \{u \in V : (v, u) \in E(\vec{G})\}$, the closed outneighborhood of v in $\vec{G}$ by $N^+_{\vec{G}}[v] = N^+_{\vec{G}}(v) \cup \{v\}$. Let $\Delta^+(\vec{G}) = \max_{v\in V} |N^+_{\vec{G}}(v)|$, $\Delta^*(G)=\min_{\vec{G}\in O(G)} \Delta^+(\vec{G})$. Let $\Pi(G)$ be the set of linear orderings of the vertices of G. For $L \in \Pi(G)$,the orientation $G_L=(V, E_L)$ of G with respect to L is obtained by setting $E_L=\{(v, u) : \{u, v\} \in E \text{ and } v>_L u\}$. The *coloring number* of G, denoted by $\mathrm{col}(G)$, is defined by

$$\mathrm{col}(G) = \min_{L\in\Pi(G)} \max_{v\in V(G)} |N^+_{G_L}[v]|.$$

This definition was motivated by the following observation. Suppose that $L \in \Pi(G)$ witnesses that $\mathrm{col}(G)\leqslant t$. If we color the vertices of G in the order L using first-fit then we will use at most t colors, since any uncolored vertex will have at most $t - 1$ colored neighbors. It follows that $\chi(G)\leqslant \mathrm{col}(G)$.

For a graph $G = (V, E)$, the game coloring number of G is defined through a *marking game*. The marking game is played by two players, Alice and Bob, with Alice playing first. At the start of the game all vertices are unmarked.

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A play by either player consists of marking an unmarked vertex. The game ends when all the vertices have been marked. Together the players create a linear order L on the vertices of G defined by $u <_L v$ if u is marked before v. The score of the game is

$$s = \max_{v \in V(G)} |N^+_{G_L}[v]|.$$

Alice's goal is to minimize the score, while Bob's goal is to maximize the score. The *game coloring number*, denoted by $\mathrm{gcol}(G)$, of G is the least s such that Alice has a strategy that results in a score of at most s. If $\mathbb{C}$ is a class of graphs then $\mathrm{gcol}(\mathbb{C}) = \max_{G \in \mathbb{C}} \mathrm{gcol}(G)$.

The *coloring game* is played like the marking game, except that instead of marking vertices the players color them from a set of colors X so that no two adjacent vertices receive the same color. Alice wins if eventually the whole graph is properly colored. Bob wins if there comes a time when all the colors have been used on the neighborhood of some uncolored vertex u. The *game chromatic number*, denoted $\chi_{\mathrm{g}}(G)$, of G is the least t such that Alice has a winning strategy in the coloring game on G using a set of t colors.

The game coloring number was first explicitly introduced by Zhu in [9] as a tool to bound the game chromatic number. It is easy to see that for any graph G, $\chi_{\mathrm{g}}(G) \leqslant \mathrm{gcol}(G)$. For the classes of interval, chordal, planar, and outerplanar graphs, which have served as benchmark problems in this subject, the best-known upper bounds for their game chromatic number are obtained by finding upper bounds for their game coloring number. The game coloring number of a graph and its generalization to oriented graphs are also of independent interests, and have been studied extensively.

In this paper, we are interested in an *asymmetric* variant of the marking game called the (a, b)-marking game. This game is played and scored like the marking game, except that on each turn Alice marks a vertices and Bob marks b vertices. (If the last vertex is marked during a player's turn, then this completes the turn.) The (a, b)-game coloring number, denoted by (a, b)-$\mathrm{gcol}(G)$, of G is the least s such that Alice has a strategy that results in a score of at most s. If $\mathbb{C}$ is a class of graphs then (a, b)-$\mathrm{gcol}(\mathbb{C}) = \max_{G \in \mathbb{C}}(a, b)$-$\mathrm{gcol}(G)$. This game was introduced in Kierstead [5] where the (a, b)-game coloring number of the class of trees is determined for all positive integers a and b. In particular it was shown that if $a < b$ then the (a, b)-game coloring number is unbounded even for the class of trees. In [6] it was shown that $(2, 1)$-$\mathrm{gcol}(G) \leqslant \frac{1}{2}t^2 + \frac{3}{2}t$, where t is the acyclic chromatic number of G.

In [7] Kierstead and Yang showed that if a is an integer and G is a graph with $\Delta^*(G) = k \leqslant a$, then $(a, 1)$-$\mathrm{gcol}(G) \leqslant 2k + 2$. This was proved by introducing the so-called *Harmonious Strategy* for Alice using to play the asymmetric marking game. It was also shown that (a, b)-$\mathrm{gcol}(G)$ is bounded on the class of graphs G with $\Delta^*(G) \leqslant k$ if and only if $k \leqslant a/b$. The classes of interval, chordal, planar, outerplanar graphs, and pseudo partial k-trees were studied in [7,8] with respect to the (a, b)-marking game.

In [1] Cai and Zhu studied the game coloring number of line graphs. A graph $G = (V, E)$ is *k-degenerate* if there is a linear order L on V whose back degree is at most k. In [1], it was shown that for the k-degenerate graphs G with maximum degree Δ, the game coloring number of the line graph of G is at most $\Delta + 3k - 1$. In particular, the game coloring number of the line graph of a planar graph is at most $\Delta + 14$; the game coloring number of the line graph of a graph with arboricity i is at most $\Delta + 6i - 4$.

In this paper, we consider the $(a, 1)$-marking games on line graphs. We will show that if G is a graph with maximum degree Δ and $\Delta^*(G) = k$, then the $(a, 1)$-game coloring number of the line graph of G is at most $\Delta + 2k + \lceil \frac{k}{a} \rceil - 1$. In particular when $a = 1$, the game coloring number of the line graph of a planar graph is at most $\Delta + 8$; the game coloring number of the line graph of a graph with arboricity i is at most $\Delta + 3i - 1$.

2. Asymmetric marking games on line graphs

For a given graph $G = (V, E)$, we use $L(G)$ to denote the line graph of G. Suppose $\vec{G}$ is an orientation of the graph G. For an edge $e = (u, v)$ in the digraph $\vec{G}$, we say u is the *tail* of e, v is the *head* of e. For a vertex $x \in V(\vec{G})$, we denote the outneighborhood of x in $L(G)$ with respect to $\vec{G}$ by $N^+_{L(G)}(x) = \{(x, v) : (x, v) \in E(\vec{G})\}$, the inneighborhood of x in $L(G)$ with respect to $\vec{G}$ by $N^-_{L(G)}(x) = \{(v, x) : (v, x) \in E(\vec{G})\}$, the neighborhood of x in $L(G)$ with respect to $\vec{G}$ by $N_{L(G)}(x) = N^+_{L(G)}(x) \cup N^-_{L(G)}(x)$. In this paper, for an edge $e = (u, v)$ in $\vec{G}$, we say the outneighborhood of e in $\vec{G}$