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A recursive construction for the dual polar spaces $DQ(2n, 2)$

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Abstract

New combinatorial constructions for the near hexagons $\mathbb{I}_3$ and $DQ(6, 2)$ in terms of ordered pairs of collinear points of the generalized quadrangle $W(2)$ were given by Sahoo [B.K. Sahoo, New constructions of two slim dense near hexagons, Discrete Math. 308 (10) (2007) 2018–2024]. Replacing $W(2)$ by an arbitrary dual polar space of type $DQ(2n, 2)$, $n \geq 2$, we obtain a generalization of these constructions. By using a construction alluded to in [B. De Bruyn, A new geometrical construction for the near hexagon with parameters $(s, t, T_2) = (2, 5, \{1, 2\})$, J. Geom. 78 (2003) 50–58.] we show that these generalized constructions give rise to near $2n$-gons which are isomorphic to $\mathbb{I}_n$ and $DQ(2n, 2)$. In this way, we obtain a recursive construction for the dual polar spaces $DQ(2n, 2)$, $n \geq 2$, different from the one given in [B.N. Cooperstein, E.E. Shult, Combinatorial construction of some near polygons, J. Combin. Theory Ser. A 78 (1997) 120–140].

Keywords: Dual polar space; Near polygon; Generalized quadrangle

1. Introduction

1.1. Elementary definitions

A *near polygon* is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$, with the property that for every point $p \in \mathcal{P}$ and every line $L \in \mathcal{L}$, there exists a unique point $\pi_L(p)$ on L nearest to p. Here, distances $\mathrm{d}(\cdot, \cdot)$ are measured in the point graph or collinearity graph Γ of $\mathcal{S}$. If d is the diameter of Γ, then the near polygon $\mathcal{S}$ is called a *near* $2d$*-gon*. A near 0-gon is a point and a near 2-gon is a line. The class of the near quadrangles coincides with the class of the so-called generalized quadrangles. A good source for information on near polygons is the recent book [6] of the author. For more background information on generalized quadrangles, we refer the reader to the book of Payne and Thas [9].

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a near polygon. If x and y are two points of $\mathcal{S}$, then we write $x \sim y$ if $\mathrm{d}(x, y) = 1$ and $x \not\sim y$ if $\mathrm{d}(x, y) \neq 1$. If X_1 and X_2 are two non-empty sets of points of $\mathcal{S}$, then $\mathrm{d}(X_1, X_2)$ denotes the minimal distance between a point of X_1 and a point of X_2. If X_1 is a singleton $\{x_1\}$, we will also write $\mathrm{d}(x_1, X_2)$ instead of $\mathrm{d}(\{x_1\}, X_2)$. For every $i \in \mathbb{Z}$ and every non-empty set X of points of $\mathcal{S}$, $\Gamma_i(X)$ denotes the set of all points y for which $\mathrm{d}(y, X) = i$. If X is a singleton $\{x\}$, we will also write $\Gamma_i(x)$ instead of $\Gamma_i(\{x\})$. We define $x^\perp := \Gamma_0(x) \cup \Gamma_1(x)$ for every point

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x of $\mathcal{S}$. If X is a set of points, then we define $X^\perp := \bigcap_{x\in X} x^\perp$ (with the convention that $X^\perp = \mathcal{P}$ if $X = \emptyset$) and $X^{\perp\perp} := (X^\perp)^\perp$.

If L_1 and L_2 are two lines of a near polygon $\mathcal{S}$, then one of the following two cases occurs (see e.g. Theorem 1.3 of [6]): (i) every point of L_1 has distance $\mathrm{d}(L_1, L_2)$ from L_2 and every point of L_2 has distance $\mathrm{d}(L_1, L_2)$ from L_1; (ii) there exist unique points $x_1 \in L_1$ and $x_2 \in L_2$ such that $\mathrm{d}(x, y) = \mathrm{d}(x, x_1) + \mathrm{d}(x_1, x_2) + \mathrm{d}(x_2, y)$ for any $x \in L_1$ and any $y \in L_2$. If case (i) occurs, then we say that L_1 and L_2 are *parallel* (notation: $L_1 \parallel L_2$).

A near polygon is called *slim* if every line is incident with precisely three points. A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbours. By Theorem 4 of Brouwer and Wilbrink [2], every two points of a dense near $2n$-gon at distance $\delta \in \{0, \ldots, n\}$ from each other are contained in a unique convex sub-(near-)2δ-gon. These convex subpolygons are called *quads* if $\delta = 2$, *hexes* if $\delta = 3$ and *maxes* if $\delta = n - 1$. The maximal distance between two points of a convex subpolygon F is called the *diameter* of F and is denoted as $\mathrm{diam}(F)$. If $X_1, X_2, \ldots, X_k$ are $k \geq 1$ objects of a dense near polygon $\mathcal{S}$ (like points or sets of points), then $\langle X_1, X_2, \ldots, X_k\rangle$ denotes the smallest convex subspace of $\mathcal{S}$ containing $X_1, X_2, \ldots, X_k$.

Let F be a convex subspace of a dense near polygon $\mathcal{S}$. F is called *big* in $\mathcal{S}$ if $F \neq \mathcal{S}$ and if every point of $\mathcal{S}$ not contained in F is collinear with a (necessarily unique) point of F. A point x of $\mathcal{S}$ is called *classical* with respect to F, if there exists a unique point $x' \in F$ such that $\mathrm{d}(x, y) = \mathrm{d}(x, x') + \mathrm{d}(x', y)$ for every point y of F. We will denote the point x' also by $\pi_F(x)$ and call it the *projection* from x on F. Every point of $\Gamma_1(F)$ is classical with respect to F. If X is a set of points of $\mathcal{S}$ which are classical with respect to F, then we define $\pi_F(X) := \{\pi_F(x) \mid x \in X\}$. F is called *classical* in $\mathcal{S}$ if every point of $\mathcal{S}$ is classical with respect to F. Every big subpolygon of $\mathcal{S}$ is classical in $\mathcal{S}$.

If F_1 and F_2 are two convex subspaces of a dense near $2d$-gon $\mathcal{S}$ with respective diameters d_1 and d_2 such that $F_1 \cap F_2 \neq \emptyset$ and F_1 is classical in $\mathcal{S}$, then the convex subspace $F_1 \cap F_2$ of $\mathcal{S}$ has diameter at least $d_1 + d_2 - d$ by Theorem 2.32 of [6].

Suppose F is a convex subpolygon of a slim dense near polygon $\mathcal{S}$. For every point x of F, we define $\mathcal{R}_F(x) := x$. If x is a point of $\mathcal{S}$ not contained in F, then we put $\mathcal{R}_F(x)$ equal to the unique point of the line $x\pi_F(x)$ different from x and $\pi_F(x)$. By Theorem 1.11 of [6], $\mathcal{R}_F$ is an automorphism of $\mathcal{S}$. $\mathcal{R}_F$ is called the *reflection about* F.

Let Q be a quad of a dense near polygon $\mathcal{S}$ and let x be a point of $\mathcal{S}$ at distance δ from Q. By Shult and Yanushka [11, Proposition 2.6], there are two possibilities. Either $\Gamma_\delta(x) \cap Q$ is a point of Q or $\Gamma_\delta(x) \cap Q$ is an *ovoid* of Q, i.e. a set of points of Q intersecting each line of Q in a unique point. In the former case, x is necessarily classical with respect to Q and we write $x \in \Gamma_{\delta,C}(Q)$. In the latter case, x is called *ovoidal with respect to* Q and we write $x \in \Gamma_{\delta,O}(Q)$.

Let $Q(2n, 2)$, $n \geq 2$, be a nonsingular parabolic quadric of $PG(2n, 2)$. Let $DQ(2n, 2)$ denote the point-line geometry whose points are the generators (= subspaces of maximal dimension $n - 1$) of $Q(2n, 2)$ and whose lines are the $(n - 2)$-dimensional subspaces of $Q(2n, 2)$, with incidence given by reverse containment. $DQ(2n, 2)$ is a so-called *dual polar space* (Cameron [3]). $DQ(2n, 2)$ is a slim dense near $2n$-gon. If α is a totally singular subspace of dimension $n - 1 - k$, $k \in \{0, \ldots, n\}$, of $Q(2n, 2)$, then the set of all generators of $Q(2n, 2)$ containing α is a convex sub-$2k$-gon of $DQ(2n, 2)$. Conversely, every convex sub-$2k$-gon of $DQ(2n, 2)$ is obtained in this way. Every convex subpolygon of $DQ(2n, 2)$ is classical in $DQ(2n, 2)$. The quads of $DQ(2n, 2)$ are isomorphic to the generalized quadrangle $W(2)$, which is the (up to isomorphisms) unique slim generalized quadrangle with three lines through each point. If x and y are two points of $DQ(2n, 2)$ at distance 2 from each other, then $\{x, y\}^{\perp\perp}$ is a set $\{x, y, z\}$ of 3 points which is contained in the quad $\langle x, y\rangle$. We call $\{x, y\}^{\perp\perp} = \{x, y, z\}$ the *hyperbolic line* of $DQ(2n, 2)$ through the points x and y. If a and b are two distinct points of $\{x, y\}^\perp$, then $\{x, y\}^\perp = \{a, b\}^{\perp\perp}$. We say that the hyperbolic lines $\{x, y\}^\perp$ and $\{x, y\}^{\perp\perp}$ of $DQ(2n, 2)$ are *orthogonal*.

Consider now a hyperplane of $PG(2n, 2)$ which intersects $Q(2n, 2)$ in a nonsingular hyperbolic quadric $Q^+(2n - 1, 2)$. The set of generators of $Q(2n, 2)$ not contained in $Q^+(2n - 1, 2)$ is a subspace of $DQ(2n, 2)$. By Brouwer et al. [1, p. 352–353], the point-line geometry induced on that subspace is a slim dense near $2n$-gon. Following the terminology of [6], we denote this near $2n$-gon by $\mathbb{I}_n$. The generalized quadrangle $\mathbb{I}_2$ is isomorphic to the (3×3)-grid. The convex subspaces of $\mathbb{I}_n$ have been studied in [6, Section 6.4]. If π is a subspace of dimension $n - 1 - k$, $k \in \{0, \ldots, n\}$, on $Q(2n, 2)$ which is not contained in $Q^+(2n - 1, 2)$ if $k \in \{0, 1\}$, then the set X_π of all generators of $Q(2n, 2)$ through π which are not contained in $Q^+(2n - 1, 2)$ is a convex sub-$2k$-gon of $\mathbb{I}_n$. Conversely, every convex sub-$2k$-gon of $\mathbb{I}_n$ is obtained in this way. If $k \geq 2$ and π is not contained in $Q^+(2n-1, 2)$, then (the point-line geometry induced on) X_π is isomorphic to $DQ(2k, 2)$. If $k \geq 2$ and π is contained in $Q^+(2n - 1, 2)$, then X_π is isomorphic to