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Note

The independence number in graphs of maximum degree three

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Abstract

We prove that a K_4-free graph G of order n, size m and maximum degree at most three has an independent set of cardinality at least $\frac{1}{7}(4n - m - \lambda - tr)$, where λ counts the number of components of G whose blocks are each either isomorphic to one of four specific graphs or edges between two of these four specific graphs and tr is the maximum number of vertex-disjoint triangles in G. Our result generalizes a bound due to Heckman and Thomas [C.C. Heckman, R. Thomas, A new proof of the independence ratio of triangle-free cubic graphs, Discrete Math. 233 (2001) 233–237].

Keywords: Independence; Triangle; Cubic graph

We consider finite simple and undirected graphs $G = (V, E)$ of order $n(G) = |V|$ and size $m(G) = |E|$. The independence number $\alpha(G)$ of G is defined as the maximum cardinality of a set of pairwise non-adjacent vertices which is called an independent set.

Our aim in the present note is to extend a result of Heckman and Thomas [6] (cf. Theorem 1) about the independence number of triangle-free graphs of maximum degree at most three to the case of graphs which may contain triangles. With their very insightful and elegant proof, Heckman and Thomas also provide a short proof for the result conjectured by Albertson, Bollobás and Tucker [1] and originally proved by Staton [9] that every triangle-free graph G of maximum degree at most three has an independent set of cardinality at least $\frac{5}{14}n(G)$ (cf. also [7]). (Note that there are exactly two connected graphs for which this bound is best-possible [2,3,5,8] and that Fraughnaugh and Locke [4] proved that every cubic triangle-free graph G has an independent set of cardinality at least $\frac{11}{30}n(G) - \frac{2}{15}$, which implies that, asymptotically, $\frac{5}{14}$ is not the correct fraction.)

In order to formulate the result of Heckman and Thomas and our extension of it we need some definitions.

A block of a graph is called *difficult* if it is isomorphic to one of the four graphs K_3, C_5, K_4^* or C_5^* in Fig. 1, i.e., it is either a triangle, or a cycle of length five, or arises by subdividing two independent edges in a K_4 twice, or arises by adding a vertex to a C_5 and joining it to three consecutive vertices of the C_5. A connected graph is called *bad* if its blocks are either difficult or are edges between difficult blocks.

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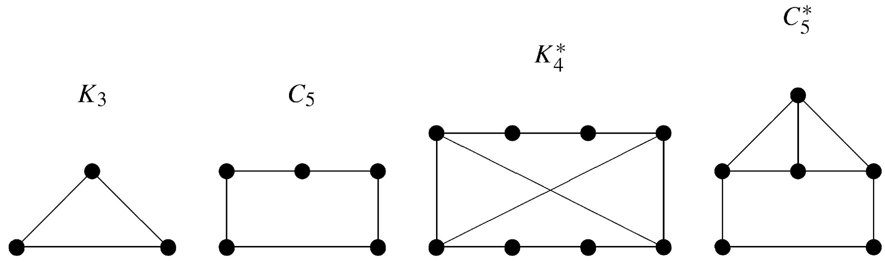

Fig. 1. Difficult blocks.

For a graph G we denote by $\lambda(G)$ the number of components of G which are bad and by $\mathrm{tr}(G)$ the maximum number of vertex-disjoint triangles in G. Note that for triangle-free graphs G our definition of $\lambda(G)$ coincides with the one given by Heckman and Thomas [6]. Furthermore, note that $\mathrm{tr}(G)$ can be computed efficiently for a graph G of maximum degree at most three as it equals exactly the number of non-trivial components of the graph formed by the edges of G which lie in a triangle of G.

Theorem 1 (*Heckman and Thomas [6]*). *Every triangle-free graph G of maximum degree at most three has an independent set of cardinality at least $\frac{1}{7}(4n(G) - m(G) - \lambda(G))$.*

Since every K_4 in a graph of maximum degree at most three must form a component and contributes exactly one to the independence number of the graph, we can restrict our attention to graphs that do not contain K_4's.

Theorem 2. *Every K_4-free graph G of maximum degree at most three has an independent set of cardinality at least $\frac{1}{7}(4n(G) - m(G) - \lambda(G) - \mathrm{tr}(G))$.*

Proof. For a graph G we denote the quantity $4n(G) - m(G) - \lambda(G) - \mathrm{tr}(G)$ by $\psi(G)$. We wish to show that $7\alpha(G) \geq \psi(G)$. For contradiction, we assume that $G = (V, E)$ is a counterexample to the statement such that $\mathrm{tr}(G)$ is smallest possible and subject to this condition the order $n(G)$ of G is smallest possible. If $\mathrm{tr}(G) = 0$, then the result follows immediately from Theorem 1. Therefore, we may assume $\mathrm{tr}(G) \geq 1$. Since $\alpha(G)$ and $\psi(G)$ are additive with respect to the components of G, we may assume that G is connected. Furthermore, we may clearly assume that $n(G) \geq 4$.

Claim 1. *Every vertex in a triangle has degree three.*

Proof of Claim 1. Let x, y and z be the vertices of a triangle. We assume that $d_G(x) = 2$. Clearly, the graph $G' = G[V \setminus \{x, y, z\}]$ is no counterexample, i.e., $7\alpha(G') \geq \psi(G')$. Since for every independent set I' of G', the set $I' \cup \{x\}$ is an independent set of G, we have $\alpha(G) \geq \alpha(G') + 1$. The triangle xyz is vertex-disjoint from all triangles in G', and so $\mathrm{tr}(G) \geq \mathrm{tr}(G') + 1$.

Suppose $\min\{d_G(y), d_G(z)\} = 2$. Then $\max\{d_G(y), d_G(z)\} = 3$, since G is not just a triangle. Furthermore, by the definition of a bad graph, we have $\lambda(G') = \lambda(G)$ and obtain

$$\begin{aligned}
7\alpha(G) &\geq 7\alpha(G') + 7 \\
&\geq \psi(G') + 7 \\
&= 4n(G') - m(G') - \lambda(G') - \mathrm{tr}(G') + 7 \\
&\geq 4(n(G) - 3) - (m(G) - 4) - \lambda(G) - (\mathrm{tr}(G) - 1) + 7 \\
&\geq \psi(G) - 12 + 4 + 1 + 7 \\
&= \psi(G),
\end{aligned}$$

which implies a contradiction. Therefore, we may assume $d_G(y) = d_G(z) = 3$. Let $N_G(y) = \{x, y', z\}$ and $N_G(z) = \{x, y, z'\}$. Regardless of whether $y' = z'$ or not, we have $\mathrm{tr}(G) \geq \mathrm{tr}(G') + 1$.

If $y' = z'$, then G' is connected, y' is a vertex of degree one in G' and thus $\lambda(G') = \lambda(G) = 0$. If $y' \neq z'$ and $\lambda(G') \geq 2$, then $\lambda(G') = 2$ and G is a bad graph itself, i.e., $\lambda(G) = 1$. Therefore, in both cases $\lambda(G') \leq \lambda(G) + 1$,