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Note

Lattice path encodings in a combinatorial proof of a differential identity

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Abstract

We specify procedures by which Łukasiewicz paths can encode combinatorial objects, such as involutions, partitions, and permutations. As application, we use these encoding procedures to give a combinatorial proof of the differential operator identity

$$\exp\left(y\left(\frac{\mathrm{d}}{\mathrm{d}x}+f(x)\right)\right)=\exp\left(\int_0^y f(t+x)\mathrm{d}t\right)\exp\left(y\frac{\mathrm{d}}{\mathrm{d}x}\right),$$

due to Stanley. Taylor's theorem is a special case of this differential identity where $f(x)=0$.

Keywords: Lattice paths; Motzkin paths; Łukasiewicz paths; Differential operators

1. Introduction

In this article, the author demonstrates a method of constructing certain well-known combinatorial objects – derangements, partitions, permutations – using weighted one-dimensional lattice paths, in a way that the number of the objects which can be constructed by a path is equal to the weight of that path. Constructions with such properties using Motzkin paths (one-dimensional excursions on non-negative integers where the allowed steps are up by one, down by one, or remain in place) was used by Flajolet [3] to derive continued fraction expansions for series involving factorial numbers, Euler numbers, Eulerian numbers, Stirling numbers of the first kind, and other quantities which could be represented by such combinatorial objects as would lend themselves to his method of construction. In contrast, the author uses Łukasiewicz paths, which generalize Motzkin paths by allowing the up steps to be arbitrarily large. The construction differs from Flajolet's even when restricted to the Motzkin paths, as demonstrated in Section 2.

The upshot of using these generalized lattice paths is that the free Łukasiewicz paths (those that are not restricted to excursions, and indeed have no restrictions on their starting and ending integers) naturally represent monomials of a differential operator of a single variable, given particular conditions on their weights. The idea of representing differential operator monomials as paths is, of course, not new: see, for example, [1,2,4]; the author shows a

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canonical representation of differential operator monomials by Łukasiewicz paths in Section 3. The author's particular construction of partitions and permutations leads to a way of representing certain combinatorial quantities like the Bell numbers and the factorial numbers in terms of differential operators.

Finally, the connection of weighted Łukasiewicz paths, on the one hand, to monomials of differential operators of one variable, and on the other hand, to permutations via construction using the same weights, allows for a combinatorial proof of a differential identity:

$$\exp\left(y\left(\frac{\mathrm{d}}{\mathrm{d}x}+f(x)\right)\right)=\exp\left(\int_0^y f(t+x)\mathrm{d}t\right)\exp\left(y\frac{\mathrm{d}}{\mathrm{d}x}\right), \tag{1}$$

with generic power series $f(x)$, due to Stanley [6] in the context of r-differential posets. Taylor's theorem, $g(x+y)=\exp(y\frac{\mathrm{d}}{\mathrm{d}x})g(x)$, is a special case of this identity, where $f(x)=0$. In Section 4, the author demonstrates the gist of the proof of the generic differential identity (1) via Proposition 5, using the special case where $f(x)=\sum_{j\geq 0}x^j=\frac{1}{1-x}$, since the idea for the general case is similar up to the weights on the steps of the paths.

2. Encoding combinatorial objects by Łukasiewicz paths

We restrict ourselves to paths on non-negative integers known as Łukasiewicz paths: paths which have no restrictions on the up steps $i\to i+j$, $j=0,1,2,\ldots$, but where the down steps must be $i\to i-1$.

Definition 1. A free Łukasiewicz path is a path on the non-negative integers composed of down steps $i\to i-1$, and up steps $i\to i+j$, where j can be any non-negative integer. We assign the weight d_i to each down step $i\to i-1$ and the weight u_i^j to each up step $i\to i+j$. The weight of a free Łukasiewicz path ending at height k is the product of the weights of its steps times x^k.

A Łukasiewicz path is a free Łukasiewicz path which begins and ends at 0, and the weight of a Łukasiewicz path is the product of the weights of its steps.

The next three subsections are examples of procedures for constructing a particular kind of combinatorial object, given a Łukasiewicz path. With respect to a particular construction procedure, we say that a path p *encodes* an object σ if σ can be constructed using p. The weight of the path p must equal to the number of objects p encodes (and, more generally, to the sum of the weights of the objects p encodes).

2.1. Motzkin paths encode involutions

A Motzkin path is a Łukasiewicz path composed only of up steps $i\to i+1$, constant steps $i\to i$, and down steps $i\to i-1$. In this example, we show how a Motzkin path of length n encodes an involution on n elements. The construction presented here differs from the one Flajolet gives in [3]. We consider each step of the path in turn. Suppose we are considering the kth step. If it is the constant step, we create a new singleton (k); if it is the up step, we create a new transposition $(k\ \text{␣})$, where we use the character ␣ – a "blank" – to denote a placeholder to be filled later; if it is a down step, we replace any one ␣ in our construction by k.

For the weight of the path to equal the number of the objects it encodes, the weights of the paths would have to be $u_i^0=1$, $u_i^1=1$, and $d_i=i$, since there is only one way to perform the construction for the up step or the constant step, but i ways to perform the construction for the down step $i\to i-1$.

We summarize the construction procedure in a table:

kth step	Construction	Weight factor
$i\to i$	Create a new singleton (k)	$u_i^0=1$
$i\to i+1$	Create a new cycle $(k\ \text{␣})$	$u_i^1=1$
$i\to i-1$	Replace any one ␣ with k	$d_i=i$

For example, the Motzkin path $p=0\to 1\to 2\to 1\to 1\to 0$ composed of five steps encodes the transposition (1 5) (2 3) (4) of five elements as follows.