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The Ramsey number $R(C_8, K_8)$ ☆

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Abstract

For two given graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer n such that for any graph G of order n, either G contains G_1 or the complement of G contains G_2. Let C_m denote a cycle of length m and K_n a complete graph of order n. We show that $R(C_8, K_8) = 50$.

Keywords: Ramsey number; Cycle; Complete graph

1. Introduction

All graphs considered in this paper are simple graphs without loops. For two given graphs G_1 and G_2, the *Ramsey number* $R(G_1, G_2)$ is the smallest integer n such that for any graph G of order n, either G contains G_1 or $\overline{G}$ contains G_2, where $\overline{G}$ is the complement of G. The *neighborhood* $N(v)$ of a vertex v is the set of vertices adjacent to v in G and $N[v] = N(v) \cup \{v\}$. The *maximum* and *minimum degree* of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. Let $V_1, V_2 \subseteq V(G)$. We use $E(V_1, V_2)$ to denote the set of the edges between V_1 and V_2. The *independence number* of a graph G is denoted by $\alpha(G)$. For $U \subseteq V(G)$, we write $\alpha(U)$ for $\alpha(G[U])$, where $G[U]$ is the subgraph induced by U in G. A *cycle* and a *path* of order n are denoted by C_n and P_n, respectively. A *clique* or a *complete graph* of order n is denoted by K_n. We use mK_n to denote the union of m vertex disjoint K_n. Let G_1 and G_2 be two given graphs, G_1+G_2 is a graph with vertex set $V = V(G_1) \cup V(G_2)$ and edge set $E = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$. A *Wheel* of order $n+1$ is $W_n = K_1 + C_n$ and W_n^- is a graph obtained from W_n by deleting a spoke from W_n. A *Book* $B_n = K_2 + \overline{K_n}$ is a graph of order $n+2$. For notations not defined here, we follow [2].

In 1978, Erdös et al. posed the following conjecture.

Conjecture (*Erdös et al. [5]*). $R(C_m, K_n) = (m-1)(n-1)+1$ *for* $m \geq n \geq 3$ *and* $(m, n) \neq (3, 3)$.

The conjecture was confirmed for $n = 3$ in early works on Ramsey theory [6,8]. Yang et al. [10] proved the conjecture for $n = 4$. Bollobás et al. [1] showed that the conjecture is true for $n = 5$. Schiermeyer [9] confirmed the conjecture for $n = 6$. Recently, Cheng et al. [3,4] solved the conjecture for $n = 7$. All the results as above support

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that the conjecture is true. In this paper, we calculate the value of the Ramsey number $R(C_8, K_8)$. The main result is the following.

Theorem 1. $R(C_8, K_8) = 50$.

2. Some lemmas

In order to prove Theorem 1, we need the following lemmas.

Lemma 1 ([3]). *Let G be a graph of order $7n-6$ $(n \geq 7)$ with $\alpha(G) \leq 7$. If G contains no C_n, then $\delta(G) \geq n-1$.*

Lemma 2 ([3]). *Let G be a graph of order $7n-6$ $(n \geq 7)$ with $\alpha(G) \leq 7$. If G contains no C_n, then G contains no W_{n-2}.*

Lemma 3 ([7]). $R(B_2, K_7) \leq 34$.

3. Proof of Theorem 1

Proof of Theorem 1. Let G be a graph of order 50. Suppose to the contrary that neither G contains a C_8 nor $\overline{G}$ contains a K_8. By Lemma 1, we have $\delta(G) \geq 7$. That is

$$G \text{ contains no } C_8. \tag{1}$$

$$1 \leq \alpha(G) \leq 7. \tag{2}$$

$$\delta(G) \geq 7. \tag{3}$$

Let $k \in \mathbf{N}$ and $4 \leq k \leq 6$. If G contains $K_1 + P_k$ as a subgraph, let $P_k = v_1 \cdots v_k$ and $V(P_k) \subseteq N(v_0)$. If G contains W_k or W_k^-, let $C = v_1 \cdots v_k$, $W_k = \{v_0\} + C$ and $W_k^- = \{v_0\} + C - \{v_0v_1\}$. In both cases, let $I = \{0, 1, \ldots, k\}$ and $S = \{v_i \mid i \in I\}$. If G contains K_k as a subgraph, let $\{v_1, \ldots, v_k\}$ be a clique. If G contains a B_{k-2}, let $v_1v_2 \in E(G)$ and $v_3, \ldots, v_k \in N(v_1) \cap N(v_2)$. In both cases, let $I = \{1, \ldots, k\}$ and $S = \{v_i \mid i \in I\}$. In all cases, set $U = V(G) - S$ and $U_i = N_U(v_i)$ for $i \in I$. By (3), $|U_i| \neq \emptyset$ for $i \in I$. If $U_i \cap U_j \neq \emptyset$ for some $i, j \in I$, let $v_{k+1} \in U_i \cap U_j$. Set $I' = I \cup \{k+1\}$, $X = S \cup \{v_{k+1}\}$, $Y = V(G) - X$ and $Y_i = N_Y(v_i)$ for $i \in I'$. If $k \leq 5$, then $Y_i \neq \emptyset$ for $i \in I'$. If $Y_i \neq \emptyset$, then for each $i \in I'$, let z_i be an arbitrary vertex in Y_i and let $Z_i = N_Y(z_i)$.

Let I be an index set, $A_i \subseteq V(G)$ for $i \in I$, and $I_1 = \{i_1, i_2, \ldots, i_k\} \subseteq I$. We say that $A_{i_1}, \ldots, A_{i_k}$ have **Property A** if

$$A_i \cap A_j = \emptyset \quad \text{for } i \in I_1,\ j \in I \text{ and } j \neq i,$$
$$\text{and } E(A_i, A_j) = \emptyset \quad \text{for } i, j \in I_1 \text{ and } i \neq j.$$

We say that $A_{i_1}, \ldots, A_{i_k}$ have **Property B** if

$$A_i \cap A_j = \emptyset \quad \text{and} \quad E(A_i, A_j) = \emptyset \quad \text{for } i, j \in I_1 \text{ and } i \neq j,$$
$$\text{and} \quad \alpha\left(\bigcup_{i \in I_1} A_i\right) = \sum_{i \in I_1} \alpha(A_i) \geq 8.$$

These notations will be used throughout the proof of Theorem 1.

In order to prove Theorem 1, we need the following claims.

Claim 1. *G contains no $K_1 + P_6$.*

Proof. Suppose, to the contrary, that G contains $K_1 + P_6$. By (1), we have $U_2 \cap U_3 = U_4 \cap U_5 = \emptyset$ and $U_i \cap U_j = \emptyset$ for $i = 1, 6$, $j \in I$ and $j \neq i$.

If $U_2 \cap U_5 \neq \emptyset$. By (1), we have $v_1v_6, v_3v_7 \notin E(G)$. By (3), $Y_i \neq \emptyset$ for $i = 1, 3, 6, 7$. By (1), we have Y_1, Y_3, Y_6, and Y_7 have Property A, thus $|Z_i| \geq 6$ and $\alpha(Z_i) \geq 2$ for $i = 1, 3, 6, 7$ by Lemma 2. By (1), Z_1, Z_3, Z_6, and Z_7 have Property B, a contradiction. Hence $U_2 \cap U_5 = \emptyset$.

If $U_0 \cap U_5 \neq \emptyset$. By Lemma 2, $v_1v_6, v_1v_7 \notin E(G)$. By (3), $Y_1, Y_6, Y_7 \neq \emptyset$. If $Y_2 = \emptyset$, then $N[v_2] = X$. By (1), $v_3v_6 \notin E(G)$, which implies that $Y_3 \neq \emptyset$. It is clear that Y_1, Y_3, Y_6, and Y_7 have Property A, thus $|Z_i| \geq 6$ and