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Note

On a property of minimal triangulations

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Abstract

A graph H has the property MT, if for all graphs G, G is H-free if and only if every minimal (chordal) triangulation of G is H-free. We show that a graph H satisfies property MT if and only if H is edgeless, H is connected and is an induced subgraph of P_5, or H has two connected components and is an induced subgraph of $2P_3$.

This completes the results of Parra and Scheffler, who have shown that MT holds for $H = P_k$, the path on k vertices, if and only if $k \leqslant 5$ [A. Parra, P. Scheffler, Characterizations and algorithmic applications of chordal graph embeddings, Discrete Applied Mathematics 79 (1997) 171–188], and of Meister, who proved that MT holds for ℓP_2, ℓ copies of a P_2, if and only if $\ell \leqslant 2$ [D. Meister, A complete characterisation of minimal triangulations of $2K_2$-free graphs, Discrete Mathematics 306 (2006) 3327–3333].

Keywords: Chordal graph; Minimal triangulation; Minimal separator

1. Introduction

Minimal triangulations of graphs have been studied for various reasons. The first $O(nm)$ time algorithm to compute a minimal triangulation of a graph dates back to 1976 [12]. A lot of the interest in minimal triangulations of graphs stems from their strong relation to the treewidth and the minimum fill-in of graphs. Research on the algorithmic complexity of those two well-known and fundamental NP-hard graph problems on particular graph classes, to a large extent carried out in the last decade of the last century, has led to interesting insights into the structure of minimal triangulations of graphs and how to use them for the design of efficient algorithms to compute the treewidth and the minimum fill-in on particular graph classes (see e.g. [2–4,8,11]). An excellent survey on the research about minimal triangulations has been provided by HEGGERNES [7] in a special issue of *Discrete Mathematics* entitled "Minimal separation and minimal triangulation" [1].

One of the early and important contributions concerning minimal triangulations of graphs in particular graph classes was given by MÖHRING. He showed that every minimal triangulation of an AT-free graph is an AT-free graph, and thus an interval graph [10]. It was soon discovered that indeed, a graph G is AT-free if and only if every minimal triangulation of G is AT-free (see e.g. [11]). In this line of research, PARRA and SCHEFFLER showed that for all $k \leqslant 5$, and every graph G, G is P_k-free if and only if every minimal triangulation of G is P_k-free, where P_k is the path on k vertices [11]. They also showed by means of an example that this property does not hold if $k \geqslant 6$ [11].

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Recently MEISTER reconsidered this property and showed among others, that for every graph G, G is $2K_2$-free if and only if every minimal triangulation of G is $2K_2$-free, where $2K_2 = 2P_2$ is the union of two copies of P_2 [9].

Here we continue this line of research. In fact, we give a characterisation (see Theorem 8) that completely settles the question which graphs H have the following property that we call property MT: *For every graph G, G is H-free if and only if all minimal triangulations of G are H-free.*

2. Preliminaries

Throughout this paper let $G = (V, E)$ be a finite simple undirected graph. The neighbourhood $N(v)$ of a vertex $v \in V$ is the set of vertices u adjacent to v. For a vertex set $S \subseteq V$, its neighbourhood is defined by $N(S) = \{u \in V \setminus S \mid N(u) \cap S \neq \varnothing\}$. The subgraph of G induced by a vertex set $S \subseteq V$ is denoted by $G[S]$. We denote the vertex set of a (connected) component of a graph by C. The corresponding maximal connected induced subgraph of G is denoted by $G[C]$. The number of connected components of a graph G is denoted by $c(G)$.

Let $H_1 + H_2$ denote the union of disjoint copies of H_1 and H_2. For $\ell \geqslant 1$, we denote by ℓG the disjoint union of ℓ copies of G. The graph P_r is an induced path on r vertices and the K_r is a complete graph on r vertices. A set $S \subseteq V$ is a clique (resp. independent set) of a graph $G = (V, E)$ if for all $u, v \in S$ the vertices u and v are adjacent (resp. non-adjacent). The maximum cardinality of an independent set of G is denoted by $\alpha(G)$.

A graph $G = (V, E)$ is *chordal* if each cycle of length at least four has a chord in G. Chordal graphs are intersection graphs of subtrees of a tree. The corresponding intersection model of a chordal graph is called a *clique tree*. The vertex set of such a clique tree T of a graph $G = (V, E)$ is the set of all maximal cliques of G and for every $v \in V$ the set of all nodes of the tree containing v is connected. For more details on chordal graphs and other graph classes we refer to [5,6].

The fundamental concepts of our paper are minimal triangulations and minimal separators. Both concepts are strongly related to chordal graphs. A *triangulation* of a graph G is a chordal graph $G^* = (V, F)$ with $E \subseteq F$. The edges in $F \setminus E$ are called *fill-edges*. G^* is a *minimal triangulation* of G if there is no triangulation of G whose fill-edges form a proper subset of $F \setminus E$. A set $S \subseteq V$ is a *separator* of G if $G - S$ is disconnected. For two vertices $u, v \in V$ of G, a separator $S \subseteq V$ is a *u, v-separator*, if u and v belong to different components of $G - S$, and S is a *minimal u, v-separator* if no proper subset S' of S is a u, v-separator. Finally S is a *minimal separator* of G if there are vertices $u, v \in V$ such that S is a minimal u, v-separator of G. We denote the set of minimal separators of a graph G by Δ_G. Note that there may be minimal separators of a graph G such that one is a proper subset of the other. We mention a well-known characterisation of chordal graphs as precisely those graphs for which each minimal separator is a clique.

We shall need some deeper understanding of minimal separators and their relations to minimal triangulations. Let $S \subseteq V$ be a separator of G, and let $G[C]$ be a connected component of $G - S$. Then, clearly $N(C) \subseteq S$. If $N(C) = S$ then $G[C]$ is a called an *S-full* component of G. It is well known that S is a minimal separator of G if G has at least two S-full components (see e.g. [6]).

Let S and T be two minimal separators. S and T are *parallel*, symbolically $S \parallel T$, if there is a component of $G - S$ containing all vertices of $T \setminus S$. Otherwise S and T *cross*, also written as $S \# T$. Notice that both $\parallel$ and # are symmetric relations.

Sets of pairwise parallel minimal separators and minimal triangulations are closely related which is nicely illustrated by the fundamental theorem of PARRA and SCHEFFLER. To formulate it we need some more notation. Let $S \in \Delta_G$ be a minimal separator. We denote by G_S the graph obtained from G by completing S, i.e., by adding an edge between every pair of non-adjacent vertices of S. If $\Gamma \subseteq \Delta_G$ then G_Γ denotes the graph obtained by completing all the minimal separators of Γ.

Theorem 1 ([11]). *Let $\Gamma \subseteq \Delta_G$ be a maximal set of pairwise parallel minimal separators of G. Then $G^* = G_\Gamma$ is a minimal triangulation of G and $\Delta_{G^*} = \Gamma$. Conversely, let G^* be a minimal triangulation of a graph G. Then Δ_{G^*} is a maximal set of pairwise parallel minimal separators of G and $G^* = G_{\Delta_{G^*}}$.*

We shall also need the following fact. Let G^* be a minimal triangulation of G. By Theorem 1, every minimal separator of G^* is also a minimal separator of G. Moreover, $G[C]$ is a connected component of $G - S$ if and only if $G^*[C]$ is a connected component of $G^* - S$.

For details and references to original work we point to [7].