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Note

On clone sets of $GF(q)$-representable matroids

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Abstract

We bound the size of a clone set in a 3-connected non-uniform $GF(q)$-representable matroid by a linear function of q. This bound is given by investigating the representability of a class of near-uniform matroids.

Keywords: Clones; Matroid; Representable

1. Introduction

Two elements of a matroid are *clones* if the map that interchanges the two elements and fixes all other elements is an automorphism of the matroid. Clones are important in the study of the representation of matroids by matrices over finite fields [4,5,7,6]. A uniform matroid contains a clone set that consists of all of its elements. For example, the matroid $U_{n-1,n}$ is representable over every field and has a clone set of size n. By contrast, we show that a non-uniform matroid that is 3-connected and $GF(q)$-representable can only contain a clone set with size bounded by a linear function in the size of the field. We also develop the connection between the problem of finding a clone set of a given size in a representable matroid and the representability of a class of near-uniform matroids. This connection is of independent interest outside of the applications given here.

Let M be a matroid on ground set $E(M)$. The notation and terminology used here follows [8] except that we use $\mathrm{si}(M)$ and $\mathrm{co}(M)$ to denote the simplification and cosimplification, respectively, of M. A *clonal class* of M is a maximal set $X \subseteq E$ such that each pair of elements of X are clones. The clonal classes of M include its set of loops, its set of coloops, each parallel class, and each series class. Such clonal classes are called *trivial clonal classes*. A *clone set* of M is a subset of a non-trivial clonal class that contains at least two elements.

The next result is due to Geelen et al. [4, Lemma 5.6]. It is followed by a result of Reid and Robbins [9, Theorem 1.2]. Together these two results determine a sharp upper bound on the size of a clone set in a 3-connected non-uniform matroid that is representable over a field with at most five elements.

Theorem 1.1. *A 2-connected binary matroid has no clone sets.*

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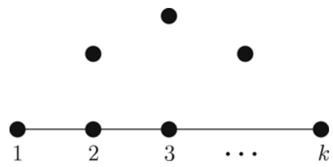

Fig. 1. Z_k.

Theorem 1.2. *For $q \in \{3, 4, 5\}$, if M is a 3-connected $GF(q)$-representable matroid that contains a clone set with at least $q - 1$ elements, then M is uniform.*

In the main result of the paper we extend the above results to all finite fields. This result can be used to quickly show that certain matroids that contain a large clone set are not representable over a particular finite field.

Theorem 1.3. *Let X be a clone set of a 3-connected non-uniform matroid M that is representable over $GF(q)$ for some $q \geq 7$.*

a. *If $q \in \{7, 8\}$, then $|X| \leq q - 2$.*
b. *If $q \in \{9, 11\}$, then $|X| \leq q - 1$.*
c. *If $q \geq 13$, then $|X| \leq 2q - 10$.*

Let Z_k be the rank-three matroid given in Fig. 1 for each $k \geq 3$. Let $q \geq 5$ be a prime power. Then Z_k is representable over $GF(q)$ precisely when $k \leq q - 2$ (Z_k can be obtained from $U_{2,k+3}$ by a Δ-Y exchange, which preserves representability). The dependent line of Z_{q-2} is a clone set of size $q-2$. So the bound given in Theorem 1.3 for $GF(7)$ and $GF(8)$ is sharp. Jakayla Robbins made the following conjecture which suggests that the upper bound given in Theorem 1.3 for $q \geq 9$ can be lowered to $q - 2$. The difficulty in obtaining such a lowered bound will be discussed at the end of the section.

Conjecture 1.4. *If a 3-connected $GF(q)$-representable matroid M has a clone set of size at least $q - 1$, then M is uniform.*

Let $U^+_{r,n}$ be the matroid obtained from $U_{r,n}$ by freely adding a point on a line. Then $U^+_{r,n}$ has a unique triangle and the deletion of any element of the triangle results in a matroid that is isomorphic to $U_{r,n}$. We call the matroid $U^+_{r,n}$ a *near-uniform* matroid. The next theorem is the key result in proving Theorem 1.3. This theorem is the second main result of the paper.

Theorem 1.5. *Let $k \geq q - 1 \geq 2$ for q a prime power. Then there exists a 3-connected non-uniform $GF(q)$-representable matroid that has a clone set of size k if and only if the matroid $U^+_{r,k+2}$ is representable over $GF(q)$, for some r with $3 \leq r \leq \lceil \frac{k+1}{2} \rceil$.*

Theorem 1.5 yields both an alternative proof for Theorem 1.2 as well as implies that Conjecture 1.4 is equivalent to the following conjecture.

Conjecture 1.6. *For each prime power q, the matroid $U^+_{r,q+1}$ is not representable over $GF(q)$ when $3 \leq r \leq \lceil \frac{q}{2} \rceil$.*

Establishing the validity of Conjecture 1.6, and hence improving the bounds for $q \geq 9$ given in Theorem 1.3, can be expected to be difficult. This is because, for example, little is known about the related question of determining the representability over finite fields of uniform matroids (see [8, Section 14.1]). This representability question of uniform matroids is of fundamental interest in projective geometry [2].

The paper is constructed as follows: we give background results on clones and representability of matroids in Section 2. In Section 3 we establish the connection between the existence of a clone set of a given size in a non-uniform representable matroid and the representability of a certain near-uniform matroid. In Section 4 we investigate the representability of these near-uniform matroids. The results of Sections 3 and 4 are then used to establish Theorem 1.3.