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Equivalence of operations with respect to discriminator clones

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Abstract

For each clone $\mathcal{C}$ on a set A there is an associated equivalence relation, called $\mathcal{C}$-equivalence, on the set of all operations on A, which relates two operations iff each one is a substitution instance of the other using operations from $\mathcal{C}$. In this paper we prove that if $\mathcal{C}$ is a discriminator clone on a finite set, then there are only finitely many $\mathcal{C}$-equivalence classes. Moreover, we show that the smallest discriminator clone is minimal with respect to this finiteness property. For discriminator clones of Boolean functions we explicitly describe the associated equivalence relations.

Keywords: Clone; Discriminator function; Minor; Subfunction; Boolean function

1. Introduction

This paper is a study of how functions on a fixed set can be classified using their substitution instances with inner functions taken from a given set of functions. In the theory of Boolean functions several variants of this idea have been employed. Harrison [5] was interested in the number of equivalence classes when n-ary Boolean functions are identified if they are substitution instances of each other with respect to the general linear group $\mathrm{GL}(n, \mathbb{F}_2)$ or the affine general linear group $\mathrm{AGL}(n, \mathbb{F}_2)$ ($\mathbb{F}_2$ is the two-element field). Wang and Williams [14] introduced classification by Boolean minors to prove that the problem of determining the threshold order of a Boolean function is NP-complete. They defined a Boolean function g to be a *minor* of another Boolean function f iff g can be obtained from f by substituting for each variable of f a variable, a negated variable, or one of the constants 0 or 1. Wang [13] characterized various classes of Boolean functions by forbidden minors. A more restrictive variant of Boolean minors, namely when negated variables are not allowed, was used in [4,15] to characterize other classes of Boolean functions by forbidden minors.

In semigroup theory, Green's relation R, when applied to transformation semigroups $\mathcal{S}$, is another occurrence of the idea of classifying functions by their substitution instances; namely, two transformations $f, g \in \mathcal{S}$ are R-related iff $f(h_1(x)) = g(x)$ and $g(h_2(x)) = f(x)$ for some $h_1, h_2 \in \mathcal{S} \cup \{\mathrm{id}\}$. Henno [6] generalized Green's relations to Menger systems (essentially, abstract clones), and described Green's relations on the clone $\mathcal{O}_A$ of all operations on A for each set A.

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The notions of $\mathcal{C}$-minor and $\mathcal{C}$-equivalence where $\mathcal{C}$ is an arbitrary clone provide a common framework for these results. If $\mathcal{C}$ is a fixed clone on a set A, and f, g are operations on A, then g is a $\mathcal{C}$-minor of f if g can be obtained from f by substituting operations from $\mathcal{C}$ for the variables of f, and g is $\mathcal{C}$-equivalent to f if f and g are both $\mathcal{C}$-minors of each other. Thus, for example, the R-relation on $\mathcal{O}_A$ described in [6] is nothing else than $\mathcal{O}_A$-equivalence, and the concepts of Boolean minor mentioned in the first paragraph are the special cases of the notion of $\mathcal{C}$-minor where $\mathcal{C}$ is the essentially unary clone of Boolean functions generated by negation and the two constants, or by the two constants only. For the least clone of Boolean functions, the essentially unary clone $\mathcal{P}$ of all projections, the $\mathcal{P}$-minor relation is investigated in [2], and the classes of Boolean functions that are closed under taking $\mathcal{P}$-minors are characterized in [3]. The latter result is extended in [10] to classes of functions on finite sets that are closed under taking $\mathcal{C}$-minors for arbitrary essentially unary clones $\mathcal{C}$. The general notions of $\mathcal{C}$-minor and $\mathcal{C}$-equivalence, as introduced at the beginning of this paragraph, first appeared in print in [7], where the first author studied the $\mathcal{C}$-minor quasiorder for clones $\mathcal{C}$ of monotone and linear operations.

The question this paper will focus on is the following:

Question. For which clones $\mathcal{C}$ on a finite set are there only finitely many $\mathcal{C}$-equivalence classes of operations?

The clones that have this property form an order filter (i.e., an upset) $\mathfrak{F}_A$ in the lattice of clones on A (see Proposition 2.3). Henno's result [6] (see Corollary 3.4) implies that $\mathcal{O}_A \in \mathfrak{F}_A$ if and only if A is finite. Thus the order filter $\mathfrak{F}_A$ is nonempty if and only if A is finite. The order filter $\mathfrak{F}_A$ is proper if $|A| > 1$, since the clone $\mathcal{P}_A$ of projections fails to belong to $\mathfrak{F}_A$. The latter statement follows from the fact that $\mathcal{P}_A$-equivalent operations have the same essential arity (i.e., depend on the same number of variables), and on a set with more than one element there exist operations of arbitrarily large essential arity.

In this paper we prove that every discriminator clone on a finite set A belongs to $\mathfrak{F}_A$. Furthermore, we show that if $|A| = 2$, then the members of $\mathfrak{F}_A$ are exactly the discriminator clones; thus in this case $\mathfrak{F}_A$ has a least member, namely the smallest discriminator clone. If $|A| > 2$, then the analogous statements are no longer true, because by a result of the first author in [8], Słupecki's clone belongs to $\mathfrak{F}_A$. Słupecki's clone consists of all operations that are either essentially unary or nonsurjective, therefore it is not a discriminator clone. Thus for finite sets with three or more elements the order filter $\mathfrak{F}_A$ remains largely unknown. However, we show that even in this case the smallest discriminator clone is a minimal member of $\mathfrak{F}_A$.

In the last section of the paper we explicitly describe the $\mathcal{C}$-equivalence and $\mathcal{C}$-minor relations for discriminator clones of Boolean functions.

2. Preliminaries

Let A be a fixed nonempty set. If n is a positive integer, then by an n-ary operation on A we mean a function $A^n \to A$, and we will refer to n as the *arity* of the operation. The set of all n-ary operations on A will be denoted by $\mathcal{O}_A^{(n)}$, and we will write $\mathcal{O}_A$ for the set of all finitary operations on A. For $1 \leq i \leq n$ the i-th n-ary *projection* is the operation $p_i^{(n)}: A^n \to A,\ (a_1, \dots, a_n) \mapsto a_i$. Every function $h: A^n \to A^m$ is uniquely determined by the m-tuple of functions $\mathbf{h} = (h_1, \dots, h_m)$ where $h_i = p_i^{(m)} \circ h: A^n \to A$ $(i = 1, \dots, m)$. In particular, $\mathbf{p}^{(n)} = (p_1^{(n)}, \dots, p_n^{(n)})$ corresponds to the identity function $A^n \to A^n$. From now on we will identify each function $h: A^n \to A^m$ with the corresponding m-tuple $\mathbf{h} = (h_1, \dots, h_m) \in (\mathcal{O}_A^{(n)})^m$ of n-ary operations. Using this convention the *composition* of functions $\mathbf{h} = (h_1, \dots, h_m): A^n \to A^m$ and $\mathbf{g} = (g_1, \dots, g_k): A^m \to A^k$ can be written as

$$\mathbf{g} \circ \mathbf{h} = (g_1 \circ \mathbf{h}, \dots, g_k \circ \mathbf{h}) = (g_1(h_1, \dots, h_m), \dots, g_k(h_1, \dots, h_m))$$

where

$$g_i(h_1, \dots, h_m)(\mathbf{a}) = g_i(h_1(\mathbf{a}), \dots, h_m(\mathbf{a})) \qquad \text{for all } \mathbf{a} \in A^n \text{ and for all } i.$$

A *clone* on A is a subset $\mathcal{C}$ of $\mathcal{O}_A$ that contains the projections and is closed under composition; more precisely, this means that for all m, n and i $(1 \leq i \leq n)$, we have $p_i^{(n)} \in \mathcal{C}$ and whenever $g \in \mathcal{C}^{(m)}$ and $\mathbf{h} \in (\mathcal{C}^{(n)})^m$ then $g \circ \mathbf{h} \in \mathcal{C}^{(n)}$. The clones on A form a complete lattice under inclusion. Therefore for each set $F \subseteq \mathcal{O}_A$ of operations there exists a smallest clone that contains F, which will be denoted by $\langle F \rangle$ and will be referred to as the *clone generated by* F.

Let $\mathcal{C}$ be a fixed clone on A. For arbitrary operations $f \in \mathcal{O}_A^{(n)}$ and $g \in \mathcal{O}_A^{(m)}$ we say that