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Local bases of primitive non-powerful signed digraphs☆

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Abstract

In 1994, Z. Li, F. Hall and C. Eschenbach extended the concept of the index of convergence from nonnegative matrices to powerful sign pattern matrices. Recently, Jiayu Shao and Lihua You studied the bases of non-powerful irreducible sign pattern matrices. In this paper, the local bases, which are generalizations of the base, of primitive non-powerful signed digraphs are introduced, and sharp bounds for local bases of primitive non-powerful signed digraphs are obtained. Furthermore, extremal digraphs are described.

Keywords: Signed digraph; Primitive; Powerful; Local base

1. Introduction and notations

The sign of a real number a, denoted by $\mathrm{sgn}(a)$, is defined to be 1, -1 or 0, according to $a > 0$, $a < 0$ or $a = 0$. An $n \times n$ sign pattern matrix $A = (a_{ij})$ has $a_{ij} \in \{1, -1, 0\}$. The sign pattern of a real matrix A, denoted by $\mathrm{sgn}(A)$, is the $(0, 1, -1)$-matrix obtained from A by replacing each entry by its sign. For a square sign pattern matrix A, notice that in the computations of (the signs of) the entries of the power A^k, the ambiguous sign may arise when -1 is added to 1. So in [4], the authors introduced a new symbol "♯" to denote the ambiguous sign. In [4], the set $\Gamma = \{0, 1, -1, \sharp\}$ is defined as the generalized sign set and the computations involving $\sharp$ are defined as follows: $(-1) + 1 = 1 + (-1) = \sharp$; $a + \sharp = \sharp + a = \sharp$ for all $a \in \Gamma$; $0 \cdot \sharp = \sharp \cdot 0 = 0$; $b \cdot \sharp = \sharp \cdot b = \sharp$ for all $b \in \Gamma \setminus 0$. An $n \times n$ generalized sign pattern matrix $A = (a_{ij})$ has $a_{ij} \in \{1, -1, 0, \sharp\}$. In this paper, we assume that all the matrix operations are operations of the matrices over the set Γ (generalized sign pattern matrices).

The graph-theoretical methods are often useful in the study of the powers of matrices, so we now introduce some graph-theoretical concepts.

A signed digraph S is a digraph where each arc of S is assigned a sign 1 or -1. A walk W in a signed digraph is a sequence of arcs $e_1, e_2, \ldots, e_k$ such that the terminal vertex of e_i is the same as the initial vertex of e_{i+1} for

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$i = 1, \ldots, k-1$. The number k is called the length of the walk W, denoted by $l(W)$. The sign of the walk W (in a signed digraph), denoted by $\mathrm{sgn}(W)$, is defined to be $\prod_{i=1}^{k} \mathrm{sgn}(e_i)$.

Two walks W_1 and W_2 in a signed digraph are called a pair of $SSSD$ walks, if they have the same initial vertex, same terminal vertex and same length, but they have different signs.

Let $A = (a_{ij})$ be a sign pattern matrix of order n. The associated digraph $D(A)$ of A (possibly with loops) is defined to be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(i, j) | a_{ij} \neq 0\}$. The associated signed digraph $S(A)$ of A is obtained from $D(A)$ by assigning the sign of a_{ij} to each arc (i, j) in $D(A)$. Clearly, we have: $(A^k)_{ij} = \sum_{W \in W_k(i,j)} \mathrm{sgn}(W)$, where $W_k(i, j)$ denotes the set of walks of length k from i to j in $S(A)$.

Definition 1.1. A square generalized sign pattern matrix A is called powerful if each power of A contains no $\sharp$ entry.

It is easy to see that a sign pattern matrix A is powerful if and only if the associated signed digraph $S(A)$ contains no pairs of $SSSD$ walks.

Definition 1.2. Let A be a generalized sign pattern matrix of order n and $A, A^2, A^3, \cdots$ be the sequence of powers of A. Suppose A^l is the first power that is repeated in the sequence. Namely, suppose l is the least positive integer such that there is a positive integer p such that

$$A^l = A^{l+p}.$$

Then l is called the generalized base (or simply base) of A, and is denoted by $l(A)$.

For convenience, we will also define the corresponding concepts for signed digraphs. Let S be a signed digraph of order n. Then there is a sign pattern matrix A of order n such that $S(A) = S$; this A is called the adjacency sign pattern matrix of S, and is denoted by $A(S)$ (or simply A). We say that S is powerful if A is powerful. Then S is powerful iff S contains no pairs of $SSSD$ walks. Also the base $l(S)$ of S is defined to be that of A, namely $l(S) = l(A)$.

A nonnegative square matrix A is primitive if some power $A^k > 0$. The least such k is called the primitive exponent (or simply exponent) of A, denoted by $\exp(A)$. For a generalized sign pattern matrix A, we use $|A|$ to denote the (0,1)-matrix obtained from A by replacing each nonzero entry by 1. For convenience, a square generalized sign pattern matrix A is called primitive if $|A|$ is primitive, and in this case we define $\exp(A) = \exp(|A|)$.

A digraph D is called a primitive digraph, if there is a positive integer k such that for all vertices x and y (not necessarily distinct) in D, there exists a walk of length k from x to y. The least such k is called the primitive exponent (or simply exponent) of D, denoted by $\exp(D)$.

Let $V(D) = \{1, 2, \ldots, n\}$. $\exp_D(i, j) :=$ the smallest integer p such that there is a walk of length t from i to j for each integer $t \geq p (1 \leq i, j \leq n)$. $\exp_D(i) := \max_{j \in V(D)} \{\exp_D(i, j)\}$ $(1 \leq i \leq n)$. Thus $\exp_D(i)$ is the smallest integer p such that there is a walk of length p (and thus of every length larger than p) from i to each vertex j of D. It follows that

$$\exp(D) = \max_{i, j \in V(D)} \{\exp_D(i, j)\} = \max_{i \in V(D)} \{\exp_D(i)\}.$$

The vertices of D can be ordered so that $\exp_D(1) \leq \exp_D(2) \leq \cdots \leq \exp_D(n)$. We call $\exp_D(k)$ the kth local exponent of D, $1 \leq k \leq n$.

For a signed digraph S, where $V(S) = \{1, 2, \ldots, n\}$, with D as its underlying digraph, we say S is primitive if D is primitive, and in this case we define $\exp(S) = \exp(D)$, $\exp_S(i, j) = \exp_D(i, j)$, and $\exp_S(i) = \exp_D(i)$.

It is well known that a sign pattern A is primitive if and only if $D(A)$ is primitive, and in this case we have $\exp(A) = \exp(D)$ (see [2]).

Proposition 1.1 ([8]). *Let S be a primitive non-powerful signed digraph. Then we have:*

(1) *There is an integer k such that there exists a pair of $SSSD$ walks of length k from each vertex x to each vertex y in S.*

(2) *If there exists a pair of $SSSD$ walks of length k from each vertex x to each vertex y, then there also exists a pair of $SSSD$ walks of length $k+1$ from each vertex u to each vertex v in S.*

(3) *The minimal such k (as in (1)) is just $l(S)$, the base of S.*