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Symmetry groups of Rosenbloom–Tsfasman spaces

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Abstract

Let $\mathbb{F}_q^{m\cdot n}$ be the vector space of $m \cdot n$-tuples over a finite field $\mathbb{F}_q$ and $P = \{1, 2, \ldots, m \cdot n\}$ a poset that is the finite union of m disjoint chains of length n. We consider on $\mathbb{F}_q^{m\cdot n}$ the poset-metric d_P introduced by Rosenbloom and Tsfasman. In this paper, we give a complete description of the symmetry group of the metric space (V, d_P).

Keywords: Poset codes; Poset metrics; Rosenbloom–Tsfasman metric; Symmetries

1. Introduction

One of the main classical problems of the coding theory is to find sets with q^k elements in $\mathbb{F}_q^n$, the space of n-tuples over the finite field $\mathbb{F}_q$, with the largest minimum distance possible. There are many possible metrics that can be defined in $\mathbb{F}_q^n$, the most common ones are the Hamming and Lee metrics.

In 1987 Harald Niederreiter generalized the classical problem of coding theory (see [7]). Brualdi, Graves and Lawrence (see [1]) also provided in 1995 a wider situation for the above problem: using partially ordered sets (posets) and defining the concept of poset-codes, they started to study codes with a poset-metric. This has been a fruitful approach, since many new perfect codes have been found with such poset metrics [3,5].

A particular instance of poset-codes and spaces are the spaces introduced by Rosenbloom and Tsfasman in 1997 [9], where the posets taken into consideration have a finite number of disjoint chains of equal size, that is, are isomorphic to a order $P = P_1 \cup P_2 \cup \cdots \cup P_m$ such that

$$P_{i+1} = \{in + 1, in + 2, \ldots, (i + 1)\, n\}$$

and

$$in + 1 < in + 2 < \cdots < (i + 1)\, n$$

are the only strict comparabilities for each $i \in \{0, 1, \ldots, m - 1\}$.

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The description of linear symmetries of a poset-space started with the study of particular poset spaces (as Lee's work on Rosenbloom–Tsfasman spaces [6]; Cho and Kim's work on crown spaces [2]; Kim's work on weak spaces [4]), until Panek, Firer, Kim and Hyun [8] gave a full description of the groups of linear symmetries of a poset space. In this work, we describe the symmetry group (not necessarily linear ones) of a poset-space that is a product of Rosenbloom–Tsfasman spaces. We hope that, as in the case of linear symmetries, the particular cases can enlighten the general one.

In the Section 2, we introduce briefly the main concepts and definitions used in this work. In the Section 3, we study the simple, but inspiring, case of posets determining a single chain (Theorem 3.1) and finally, in the last two sections, we describe the symmetry groups of Rosenbloom–Tsfasman spaces and product of such spaces (Theorems 4.1 and 5.1).

2. Poset metric spaces

Let $[n] := \{1, 2, \dots, n\}$ be a finite set with n elements and let $\leq$ be a partial order on $[n]$. We call the pair $P := ([n], \leq)$ a *poset* and say that k is *smaller than* j if $k \leq j$ and $k \neq j$. An *ideal* in $([n], \leq)$ is a subset $I \subseteq [n]$ that contains every element that is smaller than some of its elements, i.e., if $j \in I$ and $k \leq j$ then $k \in I$. Given a subset $X \subseteq [n]$, we denote by $\langle X \rangle$ the smallest ideal containing X, called the *ideal generated by* X. An order on the finite set $[n]$ is called a *linear order* or a *chain* if every two elements are comparable, that is, given $i, j \in [n]$ we have that either $i \leq j$ or $j \leq i$. In this case, n is said to be *the length* of the chain and the set can be labeled in such a way that $i_1 < i_2 < \cdots < i_n$. For the simplicity of the notation, in this situation we will always assume that the order P is defined as $1 < 2 < \cdots < n$.

Given an order $P = ([n], \leq)$ and $i, j \in [n]$, we say that $i_0, i_1, \dots, i_k$ is a *gallery (path) connecting* i and j if $i = i_0$, $j = i_k$ and for every $1 \leq l \leq k$, either $i_{l-1} \leq i_l$ or $i_l \leq i_{l-1}$. We say that i and j *can be connected* if there is a gallery connecting them. Since the concatenation of galleries is still a gallery, the above property defines an equivalence relation on $[n]$ and each equivalence class is called a *connected component* of P.

We note that every connected component of P (as every subset of $[n]$) is by itself a poset (with the induced order).

Let q be a power of a prime, $\mathbb{F}_q$ be the finite field of q elements and $V := \mathbb{F}_q^n$ the n-dimensional vector space of n-tuples over $\mathbb{F}_q$. Given $v \in V$ we denote by $v = (v_1, v_2, \dots, v_n)$ its coordinates. The poset $([n], \leq)$ induces a metric $d_P(\cdot, \cdot)$ on V, called *a poset metric* [1], defined as

$$d_P(u, v) := |\langle \mathrm{supp}(u - v) \rangle|$$

where $\mathrm{supp}(w) := \{i \in [n] : w_i \neq 0\}$ is the *support of the vector* w, and $|X|$ is the cardinality of the set X. The pair (V, d_P) is a metric space and where no ambiguity may rise, we say it is a *poset space*. The distance $\omega_P(v) := d_P(0, v)$ is called the P-*weight* of v.

A *symmetry of* (V, d_P) is a bijection $T : V \longrightarrow V$ that preserves distance:

$$d_P(u, v) = d_P(T(u), T(v))$$

for all $u, v \in V$.

The set G of symmetries of (V, d_P) is a group with the natural operation of composition of functions, and we call it the (full) *symmetry group* of (V, d_P).

The description of the full symmetry group may be of help in the study of non-linear codes. Besides other applications, linear symmetries are used to divide linear codes in equivalence classes, since they take subspace into subspace and preserve dimension and minimum distance. Symmetries, in general, may take linear codes onto non-linear ones, but preserve all metric data, such as minimal distance and weight of a code and also the generalized Wei weights, capacity of error correction and number of elements. So it is just natural to call two non-linear codes *equivalent* if one is the image of the other under a symmetry.

In [8] the group of linear symmetries of a poset space is characterized, for any given poset. In this work we will describe the full symmetry group of an important class of poset spaces, namely, those induced by posets such that every connected component is a chain. This class includes spaces over chains (linear orders) and the Rosenbloom–Tsfasman spaces, where the associated poset consists of a finite disjoint union of chains of equal length.