Available online at www.sciencedirect.com

ScienceDirect

Discrete Mathematics 309 (2009) 772–783

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Separation of clones of cooperations by cohyperidentities

K. Denecke*, K. Saengsura

Universität Potsdam, Institute of Mathematics, Am Neuen Palais, 14415 Potsdam, Germany

Received 10 August 2006; received in revised form 22 January 2008; accepted 23 January 2008
Available online 5 March 2008

Abstract

An n-ary cooperation is a mapping from a nonempty set A to the nth copower of A. A clone of cooperations is a set of cooperations which is closed under superposition and contains all injections. Coalgebras are pairs consisting of a set and a set of cooperations defined on this set. We define terms for coalgebras, coidentities and cohyperidentities. These concepts will be applied to give a new solution of the completeness problem for clones of cooperations defined on a two-element set and to separate clones of cooperations by coidentities.

Keywords: Cooperations; Coalgebras; Clones; Boolean cooperations; Coidentities; Cohyperidentities

1. Introduction

Cooperations are mappings of the form $f : A \to A^{\cup n_i}$ where $A^{\cup n_i} = \{1, \ldots, n_i\} \times A$ is the n_i-th copower of the set A in the category of sets. A clone of cooperations is a set of cooperations defined on the same set A, closed under superposition of cooperations and containing all injection cooperations $\iota_j^{n_i} : A \to A^{\cup n_i}$ defined by $a \mapsto (j, a)$ for any $a \in A$ and $1 \le j \le n_i$. A cooperation is said to be essentially n-ary if it uses precisely n different copies of A. If A is finite and $|A| = n$, then there are only finitely many clones of cooperations, since any cooperation is at most essentially n-ary. In contrast to this result, for any set A with $|A| \ge 2$ there are infinitely many clones of operations defined on A.

If $A = \{0, 1\}$, then there are exactly 12 clones of cooperations. The completeness problem for cooperations is the question of whether the clone generated by a subset $F \subseteq cO_A$ is equal to the clone cO_A of all cooperations. B. Csákány solved this problem in [1]. In [3], for operations on finite sets, a new approach to the completeness problem was proposed. In this paper we want to use similar methods for clones of cooperations. Using coidentities and cohyperidentities we will give a new solution of the completeness problem for cooperations. By the same technique we will separate any two clones of cooperations defined on $\{0, 1\}$ by coidentities. The final aim is to get an equational characterization of clones of cooperations. Different from the case of clones of operations on $\{0, 1\}$ and identities, there are clones of cooperations defined on $\{0, 1\}$ which can not be separated by coidentities.

Let A be a nonempty set and let $A^{\cup n}$ be the n-th copower of A, i.e., the union of n disjoint copies of A. Let $\underline{n} := \{0, \ldots, n-1\}$. Then $A^{\cup n} := \underline{n} \times A$ is the cartesian product of $\underline{n}$ and A; i.e., $(i, a) \in A^{\cup n}$ denotes for $0 \le i \le n-1$

* Corresponding author.
E-mail address: kdenecke@rz.uni-potsdam.de (K. Denecke).

0012-365X/$ - see front matter
doi:10.1016/j.disc.2008.01.043

the element a in the i-th copy of A. An n-ary *cooperation* is a mapping $f^A : A \to A^{\cup n}$. Each n-ary cooperation f^A is uniquely determined by a pair (f_1^A, f_2^A) of mappings $f_1^A : A \to \underline{n}$ and $f_2^A : A \to A$. This means, for every $a \in A$ we have $f^A(a) = (f_1^A(a), f_2^A(a)) \in A^{\cup n}$. We call f_1^A and f_2^A the *labelling* and the *mapping* of f^A, respectively. Unary cooperations can be regarded as unary operations defined on A. For the 16 binary cooperations on $A = \{0, 1\}$ we use the following notation

	ι_0^2	h_1	h_2	f	d	g_1	h_4	g_2
0	(0, 0)	(0, 0)	(0, 1)	(0, 1)	(0, 0)	(0, 0)	(0, 1)	(0, 1)
1	(0, 1)	(0, 0)	(0, 1)	(0, 0)	(1, 1)	(1, 0)	(1, 1)	(1, 0)

	ι_1^2	h_5	h_6	h_7	h_8	h	h_9	h_{10}
0	(1, 0)	(1, 0)	(1, 1)	(1, 1)	(1, 0)	(1, 0)	(1, 1)	(1, 1)
1	(1, 1)	(1, 0)	(1, 1)	(1, 0)	(0, 1)	(0, 0)	(0, 1)	(0, 0).

Cooperations defined on the set $\{0, 1\}$ are called *Boolean cooperations*. Let $cO_A^{(n)}$ be the set of all n-ary cooperations defined on A and let $cO_A := \bigcup_{n\geq 1} cO_A^{(n)}$ be the set of all cooperations defined on A. An *indexed coalgebra* of type τ is a pair $\mathcal{A} := (A; (f_i^{\mathcal{A}})_{i\in I})$ where $f_i^{\mathcal{A}}$ is n_i-ary and where $\tau = (n_i)_{i\in I}$ is called the *type* of the coalgebra. Coalgebras were studied first by Drbohlav [5]. These coalgebras are a special case of *functorial coalgebras*, considered for instance by Rutten in [8] which model different kinds of state-based systems, and which attracted a lot of attention during the last few years. In [1] the following superposition of cooperations was introduced. If $f^A \in cO_A^{(n)}$ and $g_0^A, \ldots, g_{n-1}^A \in cO_A^{(k)}$ then we define a k-ary cooperation $f^A[g_0^A, \ldots, g_{n-1}^A] : A \to A^{\cup k}$ by

$$a \mapsto ((g^A_{f_1^A(a)})_1(f_2^A(a)), (g^A_{f_1^A(a)})_2(f_2^A(a)))$$

for all $a \in A$. The cooperation $f^A[g_0^A, \ldots, g_{n-1}^A]$ is called the *superposition of* f^A and $g_0^A, \ldots, g_{n-1}^A$. The *injections* are special n-ary cooperations. Any set $C \subseteq cO_A$ which is closed under superposition and contains all injections is said to be a *clone of cooperations*. Clearly cO_A is a clone. Since clones can be regarded as multi-based algebras, the collection of all clones of cooperations on A forms a complete lattice, where the meet operation is the intersection. If $C \subseteq cO_A$ is a clone of cooperations and if we mean the multi-based algebra with the sequence $C^{(n)} \subseteq cO_A^{(n)}$ as universe and the superposition as operations, we will write $\underline{C}$. In [1] it was shown that for all finite sets A this lattice is finite. If A contains n elements, then any cooperation on A is determined by n-ary cooperations [1], since at most n copies of A are needed to map n elements to some copower of A. All clones of Boolean cooperations were determined in [4] (see also [1]). It is easy to check (see [4]) that the following sets of Boolean cooperations are clones:

$$C_{0c} := \bigcup_{n\geq 1}\{f \mid f \in cO^{(n)}_{\{0,1\}}(\exists i \in \underline{n}(f(0) = (i, 0)))\},$$

$$C_{1c} := \bigcup_{n\geq 1}\{f \mid f \in cO^{(n)}_{\{0,1\}}(\exists i \in \underline{n}(f(1) = (i, 1)))\},$$

$$D_c := \bigcup_{n\geq 1}\{f \mid f \in cO^{(n)}_{\{0,1\}}(f_1(a) = f_1(\neg a) \text{ and } f_2(a) = \neg f_2(\neg a), a \in \{0, 1\}) \text{ and where } \neg \text{ denotes the negation}\},$$

$$M_c := \bigcup_{n\geq 1}\{f \mid f \in cO^{(n)}_{\{0,1\}}(f_1(0) = f_1(1), f_2(0) \leq f_2(1))\},$$

$$L_c := \bigcup_{n\geq 1}\{f \mid f \in cO^{(n)}_{\{0,1\}}(\exists i \in \underline{n}(\exists a_0, a_1 \in \{0, 1\}(\forall x \in \{0, 1\}(f(x) = (i, a_0 + a_1 x)))))\},$$

$$C_2 := C_{0c} \cap C_{1c} = \bigcup_{n\geq 1}\{f \mid f \in cO^{(n)}_{\{0,1\}}(\exists i, j \in \underline{n}(f(0) = (i, 0), f(1) = (j, 1)))\},$$

$$C_3 := \bigcup_{n\geq 1}\{f \mid f \in cO^{(n)}_{\{0,1\}}(\exists i, j \in \underline{n}(f(0) = (i, 0), f(1) = (j, 0)))\} \bigcup \left(\bigcup_{n\geq 1}\{\iota_i^n \mid 0 \leq i \leq n-1\}\right),$$