Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 307 (2007) 3220–3225
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Note

The circular chromatic index of Goldberg snarks

Mohammad Ghebleh

Department of Mathematics, Simon Fraser University, Burnaby, BC, Canada

Received 11 October 2006; received in revised form 15 March 2007; accepted 16 March 2007
Available online 24 March 2007

Abstract

We determine the exact values of the circular chromatic index of the Goldberg snarks, and of a related family, the twisted Goldberg snarks.

Keywords: Edge colouring; Circular colouring; Snark; Goldberg snarks

1. Introduction

All graphs in this paper are finite and simple. Let G be a graph and $r > 2$. For every $a \in [0, r)$, let $|a|_r = \min\{|a|, r - |a|\}$. An *edge r-circular colouring*, or an *edge r–colouring* for short, of G is a function $c : E(G) \to [0, r)$ such that for any two adjacent edges e and e', $|c(e) - c(e')|_r \geqslant 1$. If G admits an edge r–colouring, then G is *edge r–colourable*. The *circular chromatic index* of G is defined by

$$\chi'_c(G) = \inf\{r \in \mathbb{R} \mid G \text{ is edge } r - \text{colourable}\}. \tag{1}$$

It is well-known, see [7] for example, that for every finite graph G, the infimum in (1) is attained, and that $\chi'_c(G)$ is rational. It is also known that for every graph G, $\chi'(G) = \lceil \chi'_c(G) \rceil$, where $\chi'(G)$ is the chromatic index of G. Hence, by Vizing's theorem, $\Delta(G) \leqslant \chi'_c(G) \leqslant \Delta(G) + 1$. It is proved in [1] that if G is a bridgeless cubic graph then

$$3 \leqslant \chi'_c(G) \leqslant 3 + \tfrac{2}{3}.$$

The upper bound is attained by the Petersen graph [7]. No bridgeless cubic graph other than the Petersen graph with the circular chromatic index greater than $3 + \frac{1}{2}$ is known. In [5] it is proved that every bridgeless cubic graph with girth at least 14 has circular chromatic index at most $3 + \frac{1}{2}$. The circular chromatic index of special classes of graphs has been of interest. For example the circular chromatic index of the flower snarks is studied in [2]. In this paper we study the circular chromatic index of the Goldberg snarks defined in [3], and of a closely related family of snarks, the twisted Goldberg snarks.

For every odd $k \geqslant 3$, the *Goldberg snark* G_k is defined as follows: $V(G_k) = \{v_j^t \mid 1 \leqslant t \leqslant k,\ 1 \leqslant j \leqslant 8\}$, and adjacencies are defined as shown in Fig. 1. The superscript t is always considered modulo k.

E-mail addresses: mghebleh@math.sfu.ca, ghebleh@gmail.com.

0012-365X/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2007.03.047

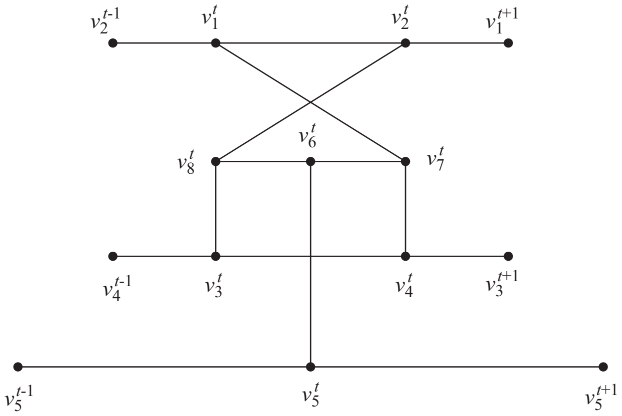

Fig. 1. Construction of Goldberg snarks.

For every odd k, we define the *twisted Goldberg snark* TG_k from G_k by replacing the edges $v_2^1v_1^2$ and $v_4^1v_3^2$ with the new edges $v_2^1v_3^2$ and $v_4^1v_1^2$. The graphs G_3 and TG_3 are shown in Fig. 4. It can be easily seen that applying more "twists" to other pairs $\{v_2^tv_1^{t+1}, v_4^tv_3^{t+1}\}$, does not produce any new graphs. Indeed an even number of twists results in G_k and an odd number of twists results in TG_k. To the author's knowledge, twisted Goldberg snarks do not appear explicitly in the literature. However, both Goldberg snarks and twisted Goldberg snarks can be obtained via the Loupekine construction [4] from the Petersen graph.

It is proved in [6] that $\chi'_c(G_k) \leqslant 3 + \frac{1}{2}$ for every odd $k \geqslant 3$. We give here the exact values of $\chi'_c(G_k)$:

Theorem 1. *For every odd $k \geqslant 3$ we have*

$$\chi'_c(G_k) = \chi'_c(TG_k) = \begin{cases} 3 + \frac{1}{3} & \text{if } k = 3, \\ 3 + \frac{1}{4} & \text{if } k \geqslant 5. \end{cases}$$

For $k = 3$, the proof relies on a computer search.

2. Goldberg snarks other than G_3

In this section we prove that for odd $k \geqslant 3$, $3 + \frac{1}{4}$ is a lower bound for the circular chromatic index of the graphs G_k and TG_k. We then show that this bound is tight when $k \geqslant 5$. Throughout this section, $k \geqslant 3$ is odd and $r = 3 + \varepsilon$ for some $\varepsilon < \frac{1}{4}$.

For any $a, b \in [0, r)$, the *r–circular interval* $[a, b]_r$ is defined as follows:

$$[a, b]_r = \begin{cases} [a, b] & \text{if } a \leqslant b, \\ [a, r) \cup [0, b] & \text{if } a > b. \end{cases}$$

For this notation, it is convenient sometimes to allow a or b to be outside the interval $[0, r)$. In that case we reduce a and b modulo r.

An easy observation used in the following proofs is that given $r < 4$, and any r–colouring c of a cubic graph G, c can be rounded down to a proper edge 3–colouring around each vertex. Namely, let e, e', e'' be the edges incident with a vertex x. Then one of $|c(e) - c(e')|_r$ and $|c(e) - c(e'')|_r$ is in the interval $[1, 1 + \varepsilon]$ and the other is in the interval $[2, 2 + \varepsilon]$, where $\varepsilon = r - 3$. By repeatedly applying this observation, we see if e' is an edge of G at distance d from the edge e, then $c(e') \in c(e) + [a, a + d\varepsilon]$ where $d \leqslant a \leqslant 2d$ is an integer.

For every $1 \leqslant t \leqslant k$, we denote the edges $v_1^tv_2^{t-1}$ and $v_3^tv_4^{t-1}$ of the Goldberg snark G_k, by e^t and f^t, respectively.

Lemma 2. *Let c be an edge r–colouring of G_k. Then for every $1 \leqslant t \leqslant k$ we have* $|c(e^t) - c(f^t)|_r \in [0, 2\varepsilon] \cup [1 - 2\varepsilon, r/2]$.