Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 309 (2009) 113–122
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Domination in bipartite graphs

Jochen Harant, Dieter Rautenbach*

Institut für Mathematik, TU Ilmenau, Postfach 100565, D-98684 Ilmenau, Germany

Received 8 December 2006; received in revised form 13 December 2007; accepted 13 December 2007
Available online 30 January 2008

Abstract

We prove that the domination number of a graph of order n and minimum degree at least 2 that does not contain cycles of length 4, 5, 7, 10 or 13 is at most $\frac{3}{8}n$. Furthermore, we derive upper bounds on the domination number of bipartite graphs of given minimum degree.

Keywords: Domination number; Cycle length; Bipartite; Probabilistic method

1. Introduction

The domination number $\gamma(G)$ of a (finite, undirected and simple) graph $G = (V, E)$ is the minimum cardinality of a set $D \subseteq V$ of vertices such that every vertex in $V \setminus D$ has a neighbour in D. This parameter is one of the most well studied in graph theory and the two volume monograph [9,10] provides an impressive account of the research related to this concept.

Fundamental results about the domination number $\gamma(G)$ are upper bounds in terms of the order n and the minimum degree δ of the graph G. Ore [14] proved that $\gamma(G) \leq \frac{n}{2}$ provided $\delta \geq 1$. For $\delta \geq 2$ and all but 7 exceptional graphs Blank [3] and McCuaig and Shepherd [13] proved

$$\gamma(G) \leq \frac{2n}{5}. \tag{1}$$

In [17] Reed proved that $\gamma(G) \leq \frac{3}{8}n$ for $\delta \geq 3$.

Bounds which are interesting for large minimum degree δ were obtained by Alon and Spencer [1], Arnautov [2] and Payan [15] who proved (see also Caro and Roditty [5,6]) that

$$\gamma(G) \leq \left(\frac{1 + \ln(\delta + 1)}{\delta + 1}\right) n. \tag{2}$$

While all these bounds hold without restricting the structure of the graph, there are several partly quite recent results [4,11,12,16,18,19] that involve conditions on the girth of the graph, i.e. the length of a shortest cycle.

* Corresponding author.
E-mail addresses: jochen.harant@tu-ilmenau.de (J. Harant), dieter.rautenbach@tu-ilmenau.de (D. Rautenbach).

0012-365X/$ - see front matter
doi:10.1016/j.disc.2007.12.051

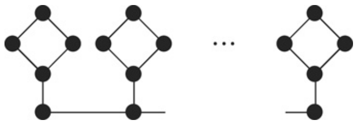

Fig. 1.

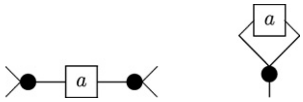

Fig. 2.

In the present paper we consider the domination number of graphs of given minimum degree under different cycle conditions related to bipartite graphs. We prove a best-possible bound on the domination number of graphs of minimum degree 2 that do not contain cycles of length 4, 5, 7, 10 or 13 and bounds on the domination number of bipartite graphs of given minimum degree.

2. Results

Graphs as in Fig. 1 show that the bound (1) [3,13] actually remains the best possible for bipartite graphs. Therefore, it makes sense to forbid cycles of length 4. Since we are eventually interested in the domination number of bipartite graphs (cf. [7,8]), we will also forbid some odd cycle lengths. Cycles of length 3 and long odd cycles can be dominated by (roughly) one-third of their vertices and do not pose a problem. Therefore, it suffices to forbid some small odd cycle length at least 5. Up to the assumption on cycles of length 10 these comments motivate the hypothesis of the following result.

Theorem 1. *If G is a graph of order n, minimum degree at least 2 and domination number γ that does not contain cycles of length 4, 5, 7, 10 or 13, then $\gamma \leq \frac{3}{8}n$.*

Proof. For contradiction, we assume that $G = (V, E)$ is a counterexample of minimum sum of order n and size. Let n and γ be as in the statement of the theorem. Since n and γ are linear with respect to the components of G, the graph G is connected. Furthermore, the set of vertices of degree at least 3 is independent.

It is easy to check the theorem for cycles and hence we can assume that G has at least one vertex of degree at least 3.

A path between vertices of degree at least 3 with a internal vertices which are all of degree 2 is called an *a-path*. Similarly, a cycle containing a vertex of degree at least 3 and a further vertices which are all of degree 2 is called an *a-loop*. See Fig. 2 for an illustration.

In what follows we will consider several times a set $V_0 \subseteq V$ of vertices with the property that $G[V \setminus V_0]$ has no vertex of degree less than 2. Note that $G[V \setminus V_0]$ satisfies the assumptions of the theorem. We will always use the following notation $n_0 = |V_0|$, $n_1 = n - n_0$, $G_0 = G[V_0]$, $G_1 = G[V \setminus V_0]$, $\gamma_0 = \gamma(G_0)$ and $\gamma_1 = \gamma(G_1)$. Note that $\gamma \leq \gamma_0 + \gamma_1$ since the union of a dominating set of G_0 and a dominating set of G_1 is a dominating set of G. Instead of a dominating set of G_0, we will sometimes consider a set $D_0 \subseteq V$ such that every vertex in V_0 is either in D_0 or adjacent to a vertex in D_0. Clearly, $\gamma \leq |D_0| + \gamma_1$.

The general approach of our proof is as follows. We consider the multigraph G' which arises from G by replacing all a-paths and a-loops by (parallel) edges and loops. Our goal is to show in a series of claims that there are no edges in G' corresponding to a-paths or a-loops with $a \equiv 0 \pmod 3$ and that the submultigraph containing the edges corresponding to a-paths or a-loops with $a \equiv 1 \pmod 3$ has maximum degree at most 2. Given these conditions, a dominating set of G containing all vertices of degree at least 3 and suitable vertices from the a-paths and a-loops will be small enough to obtain a contradiction.

Claim 1. *There is no a-path with $a \equiv 0 \pmod 3$.*