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On the existence and on the number of (k, l)-kernels in the lexicographic product of graphs

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Abstract

In [G. Hopkins, W. Staton, Some identities arising from the Fibonacci numbers of certain graphs, Fibonacci Quart. 22 (1984) 225–228.] and [I. Włoch, Generalized Fibonacci polynomial of graphs, Ars Combinatoria 68 (2003) 49–55] the total number of k-independent sets in the generalized lexicographic product of graphs was given. In this paper we study (k, l)-kernels (i.e. k-independent sets being l-dominating, simultaneously) in this product and we generalize some results from [A. Włoch, I. Włoch, The total number of maximal k-independent sets in the generalized lexicographic product of graphs, Ars Combinatoria 75 (2005) 163–170]. We give the necessary and sufficient conditions for the existence of (k, l)-kernels in it. Moreover, we construct formulas which calculate the number of all (k, l)-kernels, k-independent sets and l-dominating sets in the lexicographic product of graphs for all parameters k, l. The result concerning the total number of independent sets generalizes the Fibonacci polynomial of graphs. Also for special graphs we give some recurrence formulas.

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1. Introduction

For general concepts we refer the reader to [2,10]. By a graph G we mean a finite, undirected, connected, simple graph. $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. By a P_n we mean a graph with the vertex set $V(P_n)=\{t_1, \dots, t_n\}$ and the edge set $E(P_n)=\{\{t_i, t_{i+1}\}; i=1, \dots, n-1\}, n \geqslant 2$. Moreover, P_1 is the graph that consists of only one vertex. Let K_x denote the complete graph on x vertices, $x \geqslant 1$. Let G be a graph on $V(G)=\{t_1, \dots, t_n\}$, $n \geqslant 2$, and $h_n = (H_i)_{i\in\{1,\dots,n\}}$ be a sequence of vertex disjoint graphs on $V(H_i) = \{(t_i, y_1), \dots, (t_i, y_x)\}, x \geqslant 1$. By the generalized lexicographic product of G and $h_n = (H_i)_{i\in\{1,\dots,n\}}$ we mean the graph $G[h_n]$ such that $V(G[h_n]) = \bigcup_{i=1}^{n} V(H_i)$ and $E(G[h_n]) = \{\{(t_i, y_p), (t_j, y_q)\}; (t_i = t_j \text{and} \{(t_i, y_p), (t_i, y_q)\} \in E(H_i)) \text{or} \{t_i, t_j\} \in E(G)\}$.By H_i^c, $i = 1, \dots, n$ we will denote the copy of the graph H_i in $G[h_n]$. If $H_i = H$ for $i = 1, \dots, n$, then $G[h_n] = G[H]$, where $G[H]$ is the lexicographic product of two graphs. By $d_G(x, y)$ we denote the length of the shortest path joining vertices x and y in G.

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In [12] it has been proved:

Theorem 1 (*Włoch and Włoch [12]*). *Let* $(t_i, y_p), (t_j, y_q) \in V(G[h_n])$. *Then*

$$d_{G[h_n]}((t_i, y_p), (t_j, y_q)) = \begin{cases} d_G(t_i, t_j) & \text{for } i \neq j, \\ 1 & \text{for } i = j \text{ and } d_{H_i}(y_p, y_q) = 1, \\ 2 & \text{otherwise.} \end{cases}$$

Let $k \geqslant 2$, $l \geqslant 1$ be integers. We say that $J \subset V(G)$ is a (k, l)-kernel of a graph G if:

(1) for each $t_i, t_j \in J$, $d_G(t_i, t_j) \geqslant k$,
(2) for each $t_s \notin J$ there exists $t_i \in J$ such that $d_G(t_s, t_i) \leqslant l$.

From the definition of (k, l)-kernel it follows that if J is a (k, l)-kernel of G, then J is also a (k_0, l_0)-kernel of G where $k_0 \leqslant k$ and $l_0 \geqslant l$. If the set J satisfies condition in (1) or in (2), then we shall call it a k-independent set of G or an l-dominating set of G, respectively. We notice that 2-independent set is an independent set and 1-dominating set is a dominating set of G. In addition a subset containing only one vertex and the empty set also are k-independent sets. The set $V(G)$ is an l-dominating set of G. If an l-dominating, $l \geqslant 1$, set of G has exactly one vertex, then we shall call this vertex an l-dominating vertex of G. Moreover the l-dominating vertex of G also is a (k, l)-kernel of G, for $k \geqslant 2$.

From the definitions of k-independent set, l-dominating set and by Theorem 1 it follows:

Proposition 1. *Let* $k \geqslant 2$, $n \geqslant 2$ *be integers. A subset* $S^* \subset V(G[h_n])$ *is a* k*-independent set of* $G[h_n]$ *if and only if there exists a* k*-independent set* $S \subset V(G)$, *such that* $S^* = \bigcup_{i \in \mathscr{I}} S_i$, *where* $\mathscr{I} = \{i, t_i \in S\}$, $S_i \subset V(H_i^c)$ *and*

(a) *for* $k = 2$, S_i *is an independent set of* H_i^c,
(b) *for* $k \geqslant 3$, S_i *contains exactly one vertex from* $V(H_i^c)$

for every $i \in \mathscr{I}$.

Proposition 2. *Let* $l \geqslant 1$, $n \geqslant 2$ *be integers. A subset* $Q^* \subseteq V(G[h_n])$ *is an* l*-dominating set of* $G[h_n]$ *if and only if there exists an* l*-dominating set* $Q \subseteq V(G)$, *such that* $Q^* = \bigcup_{i \in \mathscr{I}} Q_i$, *where* $\mathscr{I} = \{i, t_i \in Q\}$, $Q_i \subseteq V(H_i^c)$ *and*

(a) *for* $l = 1$, Q_i *is a dominating set of* H_i^c *if for each* $j \in \mathscr{I}$ *and* $i \neq j$, $\{t_i, t_j\} \notin E(G)$ *or* Q_i *is a nonempty subset of* $V(H_i^c)$ *otherwise*,
(b) *for* $l \geqslant 2$, Q_i *is a nonempty subset of* $V(H_i^c)$,

for every $i \in \mathscr{I}$.

The concept of (k, l)-kernels was introduced by Kwaśnik in [5]. A (2, 1)-kernel is a kernel in Berge's sense. A (3, 1)-kernel is named as efficient dominating set and it was studied in [1]. The $(k, k-1)$-kernels, $k \geqslant 2$, were considered in [3,5,13]. In [5] it has been proved:

Theorem 2 (*Kwaśnik [5]*). *Let* $k \geqslant 2$, $l \geqslant k-1$ *be integers. Then every maximal* (*with respect to set inclusion*) k*-independent set of* G *is a* (k, l)*-kernel of* G.

The graph G has not always a (k, l)-kernel, for $k \geqslant 3$ and $l \geqslant 1$.

Theorem 3 (*Kwaśnik [5]*). *Let* $k \geqslant 2$, $l \geqslant 1$ *be integers. If the set* J *is a* (k, l)*-kernel of* G *and* $|J| \geqslant 2$, *then* $l \geqslant \frac{k-1}{2}$.

It is not easy to find a general rule when a graph G has a (k, l)-kernel. In fact there are some difficulties in finding a complete characterization of graphs having a (k, l)-kernel for $l < k-1$. For special case of k, l or for special classes