Available online at www.sciencedirect.com

ScienceDirect

Discrete Mathematics 308 (2008) 4710–4723

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Edge-colorings avoiding rainbow and monochromatic subgraphs

Maria Axenovich, Perry Iverson

Department of Mathematics, Iowa State University, USA

Received 31 May 2006; received in revised form 27 July 2007; accepted 23 August 2007
Available online 22 October 2007

Abstract

For two graphs G and H, let the mixed anti-Ramsey numbers, $\max R(n; G, H)$, $(\min R(n; G, H))$ be the maximum (minimum) number of colors used in an edge-coloring of a complete graph with n vertices having no monochromatic subgraph isomorphic to G and no totally multicolored (rainbow) subgraph isomorphic to H. These two numbers generalize the classical anti-Ramsey and Ramsey numbers, respectively.

We show that $\max R(n; G, H)$, in most cases, can be expressed in terms of vertex arboricity of H and it does not depend on the graph G. In particular, we determine $\max R(n; G, H)$ asymptotically for all graphs G and H, where G is not a star and H has vertex arboricity at least 3.

In studying $\min R(n; G, H)$ we primarily concentrate on the case when $G = H = K_3$. We find $\min R(n; K_3, K_3)$ exactly, as well as all extremal colorings. Among others, by investigating $\min R(n; K_t, K_3)$, we show that if an edge-coloring of K_n in k colors has no monochromatic K_t and no rainbow triangle, then $n \leqslant 2^{kt^2}$.

Keywords: Mixed Ramsey; Edge-coloring; Monochromatic; Totally multicolored

1. Introduction

An edge-colored graph is called monochromatic if all its edges have the same color. An edge-colored graph is called rainbow or totally multicolored if all its edges have distinct colors. For graphs G and H, we say that an edge-coloring of K_n is (G, H)-*good* if it contains neither a monochromatic copy of G nor a rainbow copy of H. The following proposition, see [19] characterizes the pairs of graphs for which (G, H)-good colorings exist for arbitrary large n.

Proposition 1 (*Jamison, West [19]*). *For any large enough n, there is a (G, H)-good coloring of $E(K_n)$ if and only if the edges of G do not induce a star and H is not a forest*.

We call a (K_s, K_t)-good coloring simply (s, t)-good. Let $\max R(n; G, H)$ $(\min R(n; G, H))$ be the maximum (minimum) number of colors in a (G, H)-good coloring of K_n.

We call these two functions mixed Ramsey numbers. They are closely related to the classical anti-Ramsey function $AR(n, H)$ and the classical multicolor Ramsey function $R_k(G)$, respectively. Here $AR(n, H)$ is defined to be the largest number of colors in an edge-coloring of K_n not containing a rainbow copy of H. This function was introduced by

E-mail addresses: axenovic@math.iastate.edu (M. Axenovich), piverson@iastate.edu (P. Iverson).

0012-365X/$ - see front matter
doi:10.1016/j.disc.2007.08.092

Erdős et al., see [13], see also [2,6,22]. The classical multicolor Ramsey function $R_k(G)$ is defined to be the smallest n such that any coloring of $E(K_n)$ in k colors contains a monochromatic copy of G, see for example [15,16]. Therefore, we see that studying $\max R(n; G, H)$ is similar to studying $AR(n, H)$ and forbidding monochromatic G. Studying $\min R(n; G, H)$ is similar to investigating $R_k(G)$ and forbidding rainbow H. Mixed Ramsey numbers are also related to various generalized Ramsey numbers. A (k, p, q)-*coloring* of $E(K_n)$ is a coloring such that each copy of K_k uses at least p and at most q colors. Thus, a $\left(k, 2, \binom{k}{2} - 1\right)$-coloring is simply a (K_k, K_k)-good coloring. The properties of (k, p, q) colorings with respect to maximum or minimum number of colors have been addressed in [3,5,8,11,24]. On the other hand, the problem of finding unavoidable rainbow H or monochromatic G in *any* coloring of K_n for n large enough, has been studied in [19], when H is a forest and G is a star, see also [18].

Observe that functions $\max R(n; G, H)$ and $\min R(n; G, H)$ are not defined for all graphs. To find all graphs for which these functions are defined, we shall need the following version of the Canonical Ramsey Theorem. Here, we say that c is a *lexical* edge-coloring of a graph F if its vertices can be ordered $v_1, \ldots, v_m$, and the colors can be renamed such that $c(v_i, v_j) = \min\{i, j\}$, for all $v_i v_j \in E(F)$.

Theorem 1 (*Deuber [10] and Erdős and Rado [12]*). *For any integers m, ℓ, r, there is an integer $n = n(m, \ell, r)$ such that any edge coloring of K_n contains either a monochromatic copy of K_m, a rainbow copy of K_r, or a lexically colored copy of K_ℓ.*

The smallest integer n, satisfying the conditions of Theorem 1 is called the Erdős–Rado number and is denoted $\mathrm{ER}(m, \ell, r)$. In general, the best bounds for symmetric Erdős–Rado numbers were provided by Lefmann and Rödl [21], in the following form:

$$2^{c_1 \ell^2} \leqslant \mathrm{ER}(\ell, \ell, \ell) \leqslant 2^{c_2 \ell^2 \log \ell}$$

for some constants c_1, c_2.

To state and prove our results we need the following definitions. The *vertex arboricity*, $a(H)$, of a graph H, is the smallest number of vertex sets partitioning $V(H)$, such that each of these sets induces a forest in H. The extremal function $\mathrm{ex}(n, H)$, for a graph H, is the largest number of edges in an n-vertex graph not containing H as a subgraph. The Turán graph $T(n, k)$ is an n-vertex complete k-partite graph with parts of almost equal sizes (different by at most one). The Turán theorem, [25], states that $\mathrm{ex}(n, K_{k+1}) = |E(T(n, k))|$. In general, the Erdős–Stone theorem, [14], states that $\mathrm{ex}(n, H) = |E(T(n, k))|(1 + \mathrm{o}(1))$, if the chromatic number of H, $\chi(H)$, is equal to $k + 1$, $k \geqslant 2$. For all other graph theoretic notions we refer the reader to [26].

Theorem 2. *Let G be a graph whose edges do not induce a star. Let H be a graph.*

(1) *If $a(H) \geqslant 3$ then*

$$\max R(n; G, H) = \frac{n^2}{2}\left(1 - \frac{1}{a(H) - 1}\right)(1 + \mathrm{o}(1)).$$

(2) *If $a(H) = 2$ then*

$$\max R(n; G, H) \leqslant c n^{2-(1/s)},$$

where $s = s(|V(G)|, |V(H)|)$.

(3) *If $a(H) = 1$ then* $\max R(n; G, H)$ *is not defined.*

Thus, in particular, Theorem 2 determines $\max R(n; G, H)$ for graphs H with vertex arboricity at least three. In the following theorem, we collect some partial results dealing with several classes of graphs H with vertex-arboricity 2. As follows from these results, $\max R(n; G, H)$ can take a wide range of values, from linear to subquadratic; the value depends heavily on the structure of H.