Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 308 (2008) 4811–4815
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Note

On endomorphism-regularity of zero-divisor graphs[☆]

Dancheng Lu[a,*], Tongsuo Wu[b]

[a] *Department of Mathematics, Suzhou University, Suzhou 215006, PR China*
[b] *Department of Mathematics, Shanghai Jiaotong University, Shanghai 200240, PR China*

Received 13 April 2006; received in revised form 17 August 2007; accepted 20 August 2007
Available online 24 September 2007

Abstract

The paper studies the following question: Given a ring R, when does the zero-divisor graph $\Gamma(R)$ have a regular endomorphism monoid? We prove if R contains at least one nontrivial idempotent, then $\Gamma(R)$ has a regular endomorphism monoid if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$; $\mathbb{Z}_2 \times \mathbb{Z}_4$; $\mathbb{Z}_2 \times (\mathbb{Z}_2[x]/(x^2))$; $F_1 \times F_2$, where F_1, F_2 are fields. In addition, we determine all positive integers n for which $\Gamma(\mathbb{Z}_n)$ has the property.

Keywords: End-regular; Zero-divisor graph; Split graph; Idempotent; Local ring

1. Introduction

Endomorphism monoids of graphs have been studied for quite some time. The main purpose of the study in this field is to reveal the relationship between graph theory and semigroup theory. One may refer to [8–11,14] for a survey along the line. The concept of regularity was first introduced by von Neumann in ring theory, where it has played an important role. Just as pointed out in [7], the most coherent part of semigroup theory at the present time is the part concerned with the structure of regular semigroups of various kinds. The following question was posed in [12]: Given a graph G, does G have a regular endomorphism monoid? It seems difficult to obtain a general answer to this question. In [10], regular endomorphism of a graph is characterized by means of idempotents. In [14,11], connected bipartite graphs and split graphs with a regular endomorphism monoid are characterized.

Another approach to reveal the relationship between algebraic structure and graph structure is using the zero-divisor graph. Let R be a commutative ring with identity 1. The zero-divisor graph of R, denoted by $\Gamma(R)$, is an undirected graph with vertices $Z(R)^*$, the set of nonzero zero-divisors of R, and for distinct $x, y \in Z(R)^*$, x is adjacent to y if and only if $xy = 0$. The zero-divisor graph of a commutative ring was first introduced by Beck in [6] and has been studied extensively in [1,2,4,5]. The zero-divisor graph concept has recently been extended to noncommutative rings in [3,13,15].

Considering the facts above, a natural question arises: Given a ring R, when does the zero-divisor graph $\Gamma(R)$ have a regular endomorphism monoid? We will study the question in the note. We prove if R contains at least one nontrivial

☆ This research is supported by National Natural Science Foundation of China (No. 10671122), partially by Collegial Natural Science Research Program of Education Department of Jiangsu Province (No. 07KJD110179) and Natural Science Foundation of Shanghai (No. 06ZR14049).
* Corresponding author.
E-mail addresses: ludancheng@suda.edu.cn (D. Lu), tswu@sjtu.edu.cn (T. Wu).

0012-365X/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2007.08.057

idempotent, then $\Gamma(R)$ has a regular endomorphism monoid if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$; $\mathbb{Z}_2 \times \mathbb{Z}_4$; $\mathbb{Z}_2 \times (\mathbb{Z}_2[x]/(x^2))$; $F_1 \times F_2$, where F_1, F_2 are fields. In addition, we determine all positive integers n for which $\Gamma(\mathbb{Z}_n)$ has the property. We also give examples of local rings (the rings without nontrivial idempotents) whose zero-divisor graphs have (resp. have not) a regular endomorphism monoid.

Throughout the note, all rings are assumed to be finite and commutative with $1 \neq 0$. If R is a ring, $Z(R)$ denotes its set of zero-divisors. A ring R is said to be *decomposable* if R can be written as $R_1 \times R_2$, where R_1 and R_2 are rings; otherwise, R is said to be *indecomposable*. Clearly, a ring is decomposable if and only if R contains at least one nontrivial idempotent. The *zero-divisor graph* of R, denoted by $\Gamma(R)$, is a graph with vertex set $Z(R)^* = Z(R) - \{0\}$, in which distinct vertices x and y are adjacent if and only if $xy = 0$. Recall a *local* ring is a ring such that $Z(R)$ is an ideal of R. It is well known that for a finite and commutative ring R, R is local if and only if R contains no nontrivial idempotents. If R is a local ring, then $u + x$ is a unit of R for any $x \in Z(R)$ and a unit u of R.

All graphs are assumed to be undirected finite graphs without loops and multiple edges. Let G and H be graphs. A mapping $f : V(G) \to V(H)$ is called a *homomorphism* from G to H if for any $a, b \in V(G)$, a is adjacent to b implies that $f(a)$ is adjacent to $f(b)$. Moreover, if f is bijective and its inverse mapping is also a homomorphism, then we call f an *isomorphism* from G to H, and in this case we say G is isomorphic to H, denoted by $G \cong H$. A homomorphism (resp. an isomorphism) from G to itself is called an *endomorphism* (resp. *automorphism*) of G. An endomorphism f is said to be *half-strong* if $f(a)$ is adjacent to $f(b)$ implies that there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that c is adjacent to d. By $\mathrm{End}(G)$, we denote the set of all the endomorphisms of G. It is well known that $\mathrm{End}(G)$ is a monoid with respect to the composition of mappings. Let x be a vertex of a graph G. The *neighborhood* (resp. *degree*) of x, denoted by $N(x)$ (resp. $d(x)$), is the set of vertices which are adjacent to x (resp. the cardinality of $N(x)$). An element a of a semigroup S is called *regular* if $a = aba$ for some $b \in S$, and S is called regular if every element in S is regular. A graph G is called *end-regular* if $\mathrm{End}(G)$ is regular.

The following results quoted from [11] will be used later.

Lemma 1.1 (*Li and Chen [11, Lemma 2.1]*). *Let G be a graph. Then every regular element in* $\mathrm{End}(G)$ *is half-strong*.

Let G be a graph. A subset U of $V(G)$ is called *complete* if every two distinct vertices of U are adjacent and is called *independent* if no two vertices in U are adjacent. A graph G is called *split* if there is a partition $V(G) = K \cup S$ of its vertex set into a complete set K and an independent set S.

Theorem 1.2 (*Li and Chen [11, Theorem 2.13]*). *Let $G(V, E)$ be a connected split graph with $V = K \cup S$ and $|K| = n$. Then G is end-regular if and only if there exists $r \in \{1, 2, \ldots, n\}$ such that $d(x) = r$ for any $x \in S$, or there exists a vertex $a \in S$ with $d(a) = n$ and there exists $r \in \{1, 2, \ldots, n-1\}$ such that $d(x) = r$ for any $x \in S - \{a\}$ (if $S - \{a\} \neq \emptyset$).*

2. Results

We begin with the following key lemma. First, we make an easy observation: if a, b are distinct vertices of a graph G such that $N(a) \subseteq N(b)$, then a is not adjacent to b, for otherwise, we have $b \in N(a) \subseteq N(b)$, a contradiction.

Lemma 2.1. *Let G be a graph. If there are pairwise distinct vertices a, b, c in G satisfying $N(c) \subsetneq N(a) \subseteq N(b)$, then G is not end-regular.*

Proof. We define a map f from $V(G)$ to itself as follows:

$$f(x) = \begin{cases} a & \text{if } x = c, \\ b & \text{if } x = a, \\ x & \text{otherwise.} \end{cases} \tag{1}$$

By the observation above, $\{a, b, c\}$ is an independent set, and it is routine to check that f is an endomorphism on G. We claim that f is not half-strong. Let x be an element in $N(a) - N(c)$. Then $f(x)$ is adjacent to $f(c)$. Note that $x \notin \{a, b, c\}$, we have $f^{-1}(x) = \{x\}$ and $f^{-1}(a) = \{c\}$. It follows that f is not half-strong since x is not adjacent to c. Now, the result follows from Lemma 1.1. □