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Note

Symmetric and resolvable λ-configurations constructed from block designs

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Abstract

A λ-configuration $(v_r, b_k)_\lambda$ is a finite incidence structure of v points and b blocks such that each block contains exactly k points, each point lies on exactly r blocks and two different points are connected by at most λ blocks. If $v = b$ and hence $r = k$, then a λ-configuration is symmetric. From any block design we construct λ-configurations. Some block designs lead to symmetric λ-configurations, and some leads to resolvable λ-configurations.

Keywords: λ-configuration; Block design; Resolvable λ-configuration

1. Introduction

Definition 1. A λ-configuration $(v_r, b_k)_\lambda$ is a finite incidence structure $(\mathscr{P}, \mathscr{B}, I)$, where $\mathscr{P}$ and $\mathscr{B}$ are disjoint sets and $I \subseteq \mathscr{P} \times \mathscr{B}$, such that

1. $|\mathscr{P}| = v$,
2. $|\mathscr{B}| = b$,
3. every element of $\mathscr{B}$ is incident with exactly k elements of $\mathscr{P}$,
4. every element of $\mathscr{P}$ is incident with exactly r elements of $\mathscr{B}$,
5. every pair of distinct elements of $\mathscr{P}$ is incident with at most λ elements of $\mathscr{B}$.

The elements of the set $\mathscr{P}$ are called points, and the elements of the set $\mathscr{B}$ are called blocks. If a point P is incident with a block x, we write PIx.

A 2-configuration is called a spatial configuration. If $v = b$ and hence $r = k$, then a λ-configuration is symmetric.

Definition 2. A parellel class or resolution class in a λ-configuration is a set of blocks that partition the point set. A resolvable λ-configuration is a λ-configuration whose blocks can be partitioned into parallel classes.

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Definition 3. Let $\mathscr{I}$ be an incidence structure with the set of points $\mathscr{P} = \{P_1, P_2, \ldots, P_v\}$ and the set of blocks $\mathscr{B} = \{x_1, x_2, \ldots, x_b\}$. The incidence matrix of $\mathscr{I}$ is a $b \times v$ matrix $M = (m_{ij})$ defined by

$$m_{ij} = \begin{cases} 1 & \text{if } P_j \text{ is incident with } x_i, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4. Let M be the incidence matrix of an incidence structure $\mathscr{I}$. Denote by M^{t} the transpose of M. The graph with adjacency matrix

$$\begin{bmatrix} 0 & M \\ M^{\mathrm{t}} & 0 \end{bmatrix}$$

is called the incidence graph of $\mathscr{I}$.

Definition 5. A (v, k, λ) block design is a λ-configuration $(v_r, b_k)_\lambda$ such that every pair of points is incident with exactly λ blocks. A $(v, 3, 1)$ block design is called a Steiner triple system.

For further basic definitions and properties of configurations and block designs we refer the reader to [1–3].

2. λ-configurations constructed from block designs

Let $\mathscr{D} = (\mathscr{P}, \mathscr{B}, I)$ be a (v, k, λ) block design. Let us define the incidence structure $\mathscr{D}_1 = (\mathscr{P}_1, \mathscr{B}_1, I_1)$ as follows:
$\mathscr{P}_1 = \{(P, x) |\ P \in \mathscr{P},\ x \in \mathscr{B},\ PIx\}$,
$\mathscr{B}_1 = \{(P, x, Q) |\ P, Q \in \mathscr{P},\ P \neq Q,\ x \in \mathscr{B},\ PIx,\ QIx\}$,
$P_1 = (P, x),\ x_1 = (\bar{P}, \bar{x}, \bar{Q}),\ P_1 I_1 x_1 \Leftrightarrow P \in \{\bar{P}, \bar{Q}\}$.

Remark 1. Let G be the incidence graph of a (v, k, λ) block design $\mathscr{D} = (\mathscr{P}, \mathscr{B}, I)$ and $\mathscr{D}_1 = (\mathscr{P}_1, \mathscr{B}_1, I_1)$ be the incidence structure defined as above. Then we can describe the incidence structure $\mathscr{D}_1$ in the following way:
$\mathscr{P}_1$ is the set of all the edges of G,
$\mathscr{B}_1$ is the set of all paths of length 2 in G with the first (and the last) vertex corresponding to a point of $\mathscr{D}$,
$P_1 \in \mathscr{P}_1,\ x_1 \in \mathscr{B}_1$, $P_1 I_1 x_1$ if and only if the union of the corresponding edge and path of length 2 is a path of length 2 or 3.

Theorem 1. *Let $\mathscr{D} = (\mathscr{P}, \mathscr{B}, I)$ be a (v, k, λ) block design and $\mathscr{D}_1 = (\mathscr{P}_1, \mathscr{B}_1, I_1)$ be the incidence structure defined as above. Then $\mathscr{D}_1$ is a $(v-1)\lambda$-configuration $(v'_{r'}, b'_{k'})_{(v-1)\lambda}$ with the following properties*:

1. $v' = vr = bk$,
2. $b' = \binom{v}{2}\lambda$,
3. $k' = 2r$,
4. $r' = (v-1)\lambda$,
5. *every pair of points is incident with exactly $(v-1)\lambda$ or λ blocks*,
6. *every pair of blocks is incident with exactly $2r$, r or 0 points.*

Proof. It is obvious that $v' = vr = bk$. Two points of the design $\mathscr{D}$ determine λ blocks in $\mathscr{D}_1$, since every pair of points in (v, k, λ) block design is incident with exactly λ blocks. Therefore $b' = \binom{v}{2}\lambda$.

If a block x_1 in $\mathscr{D}_1$ corresponds to an ordered triple (P, x, Q), then x_1 is incident with points which correspond to (P, y) or (Q, z), $y, z \in \mathscr{B}$. Since the points P and Q in $\mathscr{D}$ are incident with r blocks, x_1 is incident with $2r$ points.

A point $P_1 \in \mathscr{P}_1$ corresponding to an ordered pair (P, x) is incident with blocks which correspond to (P, y, Q), $Q \in \mathscr{P}_1$, $y \in \mathscr{B}_1$. We can choose Q in $v - 1$ ways, and each pair P, Q is incident with exactly λ blocks.

Let P_1 be a point in $\mathscr{P}_1$ which corresponds to an ordered pair (P, x), $R_1 \in \mathscr{P}_1$ corresponds to (P, y) and $Q_1 \in \mathscr{P}_1$ corresponds to (Q, z), $Q \neq P$. A block $x_1 \in \mathscr{B}_1$ is incident with P_1 if and only if it is incident with R_1. On the other