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On the signed edge domination number of graphs

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ABSTRACT

Let $\gamma_s'(G)$ be the signed edge domination number of G. In 2006, Xu conjectured that: for any 2-connected graph G of order $n(n \geq 2)$, $\gamma_s'(G) \geq 1$. In this article we show that this conjecture is not true. More precisely, we show that for any positive integer m, there exists an m-connected graph G such that $\gamma_s'(G) \leq -\frac{m}{6}|V(G)|$. Also for every two natural numbers m and n, we determine $\gamma_s'(K_{m,n})$, where $K_{m,n}$ is the complete bipartite graph with part sizes m and n.

0. Introduction

In this paper all of the graphs that we consider are finite, simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The *order* of G denotes the number of vertices of G. For any $v \in V(G)$, $d(v)$ is the degree of v and $E(v)$ is the set of all edges incident with v. If $e = uv \in E(G)$, then we put $N[e] = \{u'v' \in E(G) | u' = u \text{ or } v' = v\}$. Let G be a graph and $f : E(G) \longrightarrow \{-1, 1\}$ be a function. For every vertex v, we define $s_v = \sum_{e \in E(v)} f(e)$. We denote the complete bipartite graph with two parts of sizes m and n, by $K_{m,n}$. Also we denote the cycle of order n, by C_n. In [4] the signed edge domination function of graphs was introduced as follows:

Let $G = (V(G), E(G))$ be a non-empty graph. A function $f : E(G) \longrightarrow \{-1, 1\}$ is called a *signed edge domination function* (SEDF) of G if $\sum_{e' \in N[e]} f(e') \geq 1$, for every $e \in E(G)$. The *signed edge domination number* of G is defined as,

$$\gamma_s'(G) = \min \left\{ \sum_{e \in E(G)} f(e) \mid f \text{ is an SEDF of } G \right\}.$$

Several papers have been published on lower bounds and upper bounds of the signed edge domination number of graphs, for instance, see [2–5]. In [2], Xu posed the following conjecture:

For any 2-connected graph G of order $n(n \geq 2)$, $\gamma_s'(G) \geq 1$.

In the first section we give some counterexamples to this conjecture by showing that for any natural number m, there exists an m-connected graph G such that $\gamma_s'(G) \leq -\frac{m}{6}|V(G)|$. For any natural number k, let $g(k) = \min\{\gamma_s'(G) \mid |V(G)| = k\}$. In [2] the following problem was posed:

Determine the exact value of $g(k)$ for every positive integer k. In Section 1, it is shown that for every natural number k, $k \geq 12$, $g(k) \leq \frac{-(k-8)^2}{72}$.

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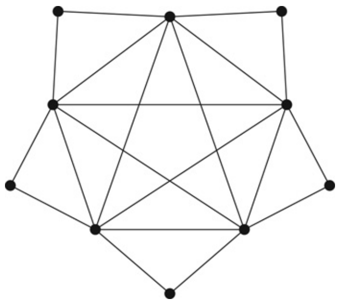

Fig. 1. A 2-connected $L_{(2,1)}$-graph with $\gamma_s' < 1$.

1. Counterexamples to a conjecture

In this section we present some counterexamples to a conjecture that appeared in [2]. We start this section by the following simple lemma and leave the proof to the reader.

Lemma 1. *Let $f : E(G) \longrightarrow \{-1, 1\}$ be a function. Then f is an* SEDF *of G, if and only if for any edge $e = uv$, $s_u + s_v - f(e) \geq 1$. Moreover, if f is an* SEDF, *then $s_u + s_v \geq 0$.*

An $L_{(m,n)}$-graph G is a graph of order $(n+1)(mn+m+1)$, whose vertices can be partitioned into $n+1$ subsets $V_1, \ldots, V_{n+1}$ such that:

(i) The induced subgraph on V_1 is the complete graph K_{mn+m+1}.
(ii) The induced subgraph on V_i, $2 \leq i \leq n+1$ is the complement of K_{mn+m+1}.
(iii) For every i, $2 \leq i \leq n+1$, all edges between V_1 and V_i form m disjoint matchings of size $mn+m+1$.
(iv) There is no edge between V_i and V_j for any i, j, $2 \leq i < j \leq n+1$.

It is well-known that for any natural number r, the edge chromatic number of $K_{r,r}$ is r, see Theorem 6 of [1, p. 93]. Thus for every pair of natural numbers m and n, there is an $L_{(m,n)}$-graph.

Theorem 1. *Let m and n be two natural numbers. Then for every $L_{(m,n)}$-graph G, we have,*

$$\gamma_s'(G) \leq \frac{(mn+m+1)(m-mn)}{2}.$$

Proof. To prove the inequality we provide an SEDF for G, say f, such that,

$$\sum_{e \in E(G)} f(e) = \frac{(mn+m+1)(m-mn)}{2}.$$

Define $f(e) = 1$, if both end points of e are contained in V_1, and $f(e) = -1$, otherwise. We find,

$$\sum_{e \in E(G)} f(e) = \frac{(mn+m+1)(mn+m)}{2} - (mn+m+1)mn$$
$$= \frac{(mn+m+1)(m-mn)}{2}.$$

It can be easily verified that for every $v \in V_1$, $s_v = m$, and for every $v \in V(G) \setminus V_1$, $s_v = -m$. Now, Lemma 1 yields that f is an SEDF for G. □

Example 1. Consider the $L_{(2,1)}$-graph G shown in Fig. 1. The graph clearly has perfect matching; and by applying Lemma 1 to the edges of this matching we may conclude that for every SEDF f of this graph, $\sum_{e \in E(G)} f(e) = \frac{1}{2} \sum_{v \in V(G)} s_v \geq 0$, and hence $\gamma_s'(G) \geq 0$. But it follows from Theorem 1 that $\gamma_s'(G) \leq 0$. Consequently, $\gamma_s'(G) = 0$ and the bound in Theorem 1 is sharp for this graph.

In [2], Xu conjectured that for any 2-connected graph G of order n $(n \geq 2)$, $\gamma_s'(G) \geq 1$. The next theorem shows that conjecture fails.

Theorem 2. *For any natural number m, there exists an m-connected graph G such that $\gamma_s'(G) \leq -\frac{m}{6}|V(G)|$.*

Proof. First we claim that for each pair of natural numbers m and n, every $L_{(m,n)}$-graph is an m-connected graph. To see this we note that if one omits at most $m-1$ vertices of an $L_{(m,n)}$-graph, then some vertices of V_1 remain (because $|V_1| = mn+m+1$) and since the degree of each vertex of V_i, $2 \leq i \leq n+1$ is m, the claim is proved.