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Long cycles in graphs without hamiltonian paths

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Abstract

For a graph G, $p(G)$ and $c(G)$ denote the order of a longest path and a longest cycle of G, respectively. Bondy and Locke [J.A. Bondy, S.C. Locke, Relative length of paths and cycles in 3-connected graphs, Discrete Math. 33 (1981) 111–122] consider the gap between $p(G)$ and $c(G)$ in 3-connected graphs G. Starting with this result, there are many results appeared in this context, see [H. Enomoto, J. van den Heuvel, A. Kaneko, A. Saito, Relative length of long paths and cycles in graphs with large degree sums, J. Graph Theory 20 (1995) 213–225; M. Lu, H. Liu, F. Tian, Relative length of longest paths and cycles in graphs, Graphs Combin. 23 (2007) 433–443; K. Ozeki, M. Tsugaki, T. Yamashita, On relative length of longest paths and cycles, preprint; I. Schiermeyer, M. Tewes, Longest paths and longest cycles in graphs with large degree sums, Graphs Combin. 18 (2002) 633–643]. In this paper, we investigate graphs G with $p(G) - c(G)$ at most 1 or at most 2, but with no hamiltonian paths. Let G be a 2-connected graph of order n, which has no hamiltonian paths. We show two results as follows: (i) if $\sigma_4(G) \geq \frac{1}{3}(4n-2)$, then $p(G) - c(G) \leq 1$, and (ii) if $\sigma_4(G) \geq n+3$, then $p(G) - c(G) \leq 2$.

Keywords: Relative length; Longest path; Longest cycle

1. Introduction

A well-known result of Bondy and Locke [1] says that if a 3-connected graph has a path of length k, then it has a cycle of length at least $2k/5$. So in 3-connected graphs, if there is a long path, then there is a long cycle too. Let $p(G)$ and $c(G)$ be the order of a longest path and a longest cycle, respectively. For a positive integer k, if there exists an independent set of order k, then we let $\sigma_k(G)$ denote the minimum degree sum of an independent set of k vertices of G; otherwise we let $\sigma_k(G) = +\infty$.

In this paper, we are interested in the difference diff(G) between $p(G)$ and $c(G)$, that is, $\text{diff}(G) := p(G) - c(G)$. In particular, we are interested in graphs that have a small gap between $p(G)$ and $c(G)$. In [2], Enomoto, van den Heuvel, Kaneko and Saito proved the following theorems:

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Theorem 1 (*Enomoto et al. [2]*). *Let G be a* 2*-connected graph on n vertices. If* $\sigma_3(G) \geq n+2$*, then* $\mathrm{diff}(G) \leq 1$.

Theorem 2 (*Enomoto et al. [2]*). *Let G be a connected graph on n vertices. If* $\sigma_3(G) \geq n$*, then either* $\mathrm{diff}(G) \leq 1$ *or G has a hamiltonian path.*

In [4], Li, Saito and Schelp considered the concerning the property "$\mathrm{diff}(G) \leq 1$" and a σ_4 condition. They proved that if G is a 3-connected graph of order n with $\sigma_4(G) \geq \frac{3}{2}n+1$, then $\mathrm{diff}(G) \leq 1$. Also they conjectured that the sharp coefficient of n is $\frac{4}{3}$. Lu, Liu and Tian gave a sharp bound on the $\sigma_4(G)$ condition.

Theorem 3 (*Lu et al. [3]*). *Let G be a* 3*-connected graph on n vertices. If* $\sigma_4(G) \geq \frac{1}{3}(4n+5)$*, then* $\mathrm{diff}(G) \leq 1$.

Theorems 1 and 2 say that the connectivity and degree sum condition can be weakened for graphs without hamiltonian paths. Therefore, one might expect that the conditions in Theorem 3 can be weakened for graphs without hamiltonian paths. As an answer to this expectation, we prove the following theorem.

Theorem 4. *Let G be a* 2*-connected graph on n vertices. If* $\sigma_4(G) \geq \frac{1}{3}(4n-2)$*, then either* $\mathrm{diff}(G) \leq 1$ *or G has a hamiltonian path.*

Recently, the second author, Tsugaki and the third author showed the result on a σ_4 condition.

Theorem 5 (*[5]*). *Let G be a* 3*-connected graph on n vertices. If* $\sigma_4(G) \geq n+6$*, then* $\mathrm{diff}(G) \leq 2$.

On the other hand, in 2002, Schiermeyer and Tewes [6] investigated the relation between $\sigma_4(G)$ and $\mathrm{diff}(G) \leq 2$ in a 2-connected graph G. A path P of a graph G is said to be dominating if $V(G-P)$ is an independent set.

Theorem 6 (*Schiermeyer and Tewes [6]*). *Let G be a* 2*-connected graph on n vertices. If* $\sigma_4(G) \geq n+3$*, then either* $\mathrm{diff}(G) \leq 2$ *or every longest path in G is dominating.*

However, considering the relations between Theorems 1 and 2 and between Theorems 3 and 4, the conclusion of Theorem 6 seems to be weak. Therefore, we give an improvement of Theorem 6.

Theorem 7. *Let G be a* 2*-connected graph on n vertices. If* $\sigma_4(G) \geq n+3$ *then either* $\mathrm{diff}(G) \leq 2$ *or G has a hamiltonian path.*

The degree sum bounds of Theorems 4 and 7 are best possible. Let m be an integer with $m \geq 2$ and $G_1 := K_m + (K_1 \cup (m+1)K_2)$. Then $\sigma_4(G_1) = m + 3(m+1) = \frac{1}{3}(4n-3)$ and neither $\mathrm{diff}(G_1) \leq 1$ nor G_1 has a hamiltonian path. On the other hand, let $G_2 := K_m + (K_1 \cup (m+1)K_3)$. Then $\sigma_4(G_2) = m + 3(m+2) = n+2$ and neither $\mathrm{diff}(G_2) \leq 2$ nor G_2 has a hamiltonian path.

We prove Theorems 4 and 7 simultaneously. For that purpose, we shall define *endable vertex* and show several claims in Section 2. In the proof of Theorems 4 and 7, we divide into two cases. Case 1 and Case 2 are discussed in Sections 3 and 4, respectively. We shall use the following lemma.

Lemma 8 (*Enomoto et al. [2]*). *Suppose that G is a graph of order n with* $\mathrm{diff}(G) \geq 2$*. Let P is a longest path in G and let* $x, y \in V(G)$ *be end-vertices of P. If there exists* $z \in V(G-P)$*, then* $d_G(x) + d_G(y) + d_G(z) \leq n-1$.

For standard graph-theoretical terminology not explained in this paper, we refer the reader to [7]. We denote by $N_G(x)$ the neighbourhood of a vertex x in a graph G. For a subgraph H of G and a vertex $x \in V(G) \setminus V(H)$, we denote $N_H(x) := N_G(x) \cap V(H)$, and $d_H(x) := |N_H(x)|$. Furthermore, for a subgraph H of G and $X \subset V(G) \setminus V(H)$, we write $N_H(X) := \bigcup_{x \in V(X)} N_H(x)$. If there is no fear of confusion, we often identify a subgraph H of a graph G with its vertex set $V(H)$. We write a cycle (or a path) C with a given orientation by $\overrightarrow{C}$. Let C be a cycle or a path. For $x, y \in V(C)$, we denote by $x\overrightarrow{C}y$ a path from x to y on $\overrightarrow{C}$. The reverse sequence of $x\overrightarrow{C}y$ is denoted by $y\overleftarrow{C}x$. For $x \in V(C)$, we denote the h-th successor and the h-th predecessor of x on $\overrightarrow{C}$ by x^{+h} and x^{-h}, respectively. We abbreviate x^{+1} and x^{-1} by x^+ and x^-, respectively. For $X \subset V(C)$, we define $X^{+h} := \{x^{+h} : x \in X\}$ and $X^{-h} := \{x^{-h} : x \in X\}$, and abbreviate X^{+1} and X^{-1} by X^+ and X^-, respectively.