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Restricted matching in graphs of small genus

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Abstract

A graph G with at least $2n+2$ vertices is said to be *n-extendable* if every set of n disjoint edges in G extends to (i.e., is a subset of) a perfect matching. More generally, a graph is said to have property $E(m,n)$ if, for every matching M of size m and every matching N of size n in G such that $M \cap N = \emptyset$, there is a perfect matching F in G such that $M \subseteq F$, but $F \cap N = \emptyset$. G is said to have property $E(0,0)$ if it has a perfect matching. The study of the properties $E(m,n)$ is referred to as the study of *restricted* matching extension.

In [M. Porteous, R. Aldred, Matching extensions with prescribed and forbidden edges, Australas. J. Combin. 13 (1996) 163–174; M. Porteous, Generalizing matching extensions, M.A. Thesis, University of Otago, 1995; A. McGregor-Macdonald, The $E(m,n)$ property, M.Sc. Thesis, University of Otago, 2000], Porteous and Aldred, Porteous and McGregor-Macdonald, respectively, studied the possible implications among the properties $E(m,n)$ for various values of m and n. In an earlier paper [R.E.L. Aldred, Michael D. Plummer, On restricted matching extension in planar graphs, in: 17th British Combinatorial Conference (Canterbury 1999), Discrete Math. 231 (2001) 73–79], the present authors completely determined which of the various properties $E(m,n)$ always hold, sometimes hold and never hold for graphs embedded in the plane. In the present paper, we do the same for embeddings in the projective plane, the torus and the Klein bottle.

Keywords: Genus; Restricted matching; Klein bottle; Projective plane; Torus

1. Introduction

A *matching* in a graph G is any set of independent edges; i.e., edges no two of which have a vertex in common. A matching is *perfect* if it spans all vertices of G. A matching M in graph G is *extendable* if it is a subset of some perfect matching in G. Graph G is said to be *n-extendable* if $|V(G)| \geq 2n+2$ and every matching of size n in G extends to (i.e., is a subset of) a perfect matching in G. More generally, graph G is said to have property $E(m,n)$ (or more briefly, is said to be $E(m,n)$) if $|V(G)| \geq 2(m+n+1)$ and for every matching M of size m and every matching N of size n in G such that $M \cap N = \emptyset$, there is a perfect matching F in G such that $M \subseteq F$, but $N \cap F = \emptyset$. In [13,12,6], Porteous and Aldred, Porteous and McGregor-Macdonald, respectively, studied the possible implications among the properties $E(m,n)$ for various values of m and n. That portion of the lattice of implications (and non-implications) which will be pertinent to the present paper is summarized in Fig. 1.1.

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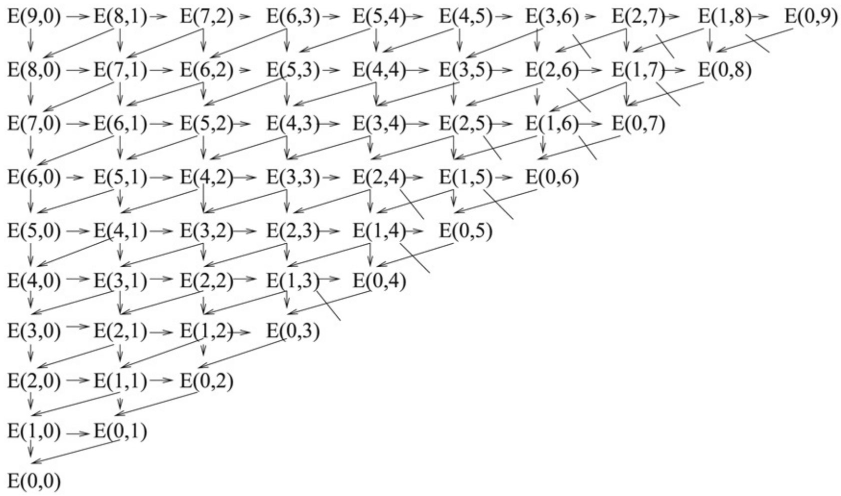

Fig. 1.1. A partial lattice of implications for the $E(m, n)$ property.

Lemma 1.1 (*[13, Corollary 2.2]*). *If graph G is $E(m, n)$, then G is $(m + 1)$-connected.*

An embedding of a graph G on the surface Σ is said to be *2-cell* if every face of the embedding is homeomorphic to a disc. For 2-cell embeddings, we have Euler's formula:

Theorem 1.2. *If G is a 2-cell embedded on the surface Σ having genus g (resp. non-orientable genus $\overline{g}$) and if the embedded G has $|V(G)| = p$ vertices, $|E(G)| = q$ edges and $|F(G)| = f$ faces, then $p - q + f = 2 - 2g$ (resp. $p - q + f = 2 - \overline{g}$).*

The following two results are of paramount importance when working with minimal embeddings. The first is due to Youngs [18] and the second to Parsons, Pica, Pisanski and Ventre [9].

Theorem 1.3. *Every minimal orientable embedding of a graph G is a 2-cell.*

Theorem 1.4. *Every graph G has a minimal non-orientable embedding which is a 2-cell.*

We shall also make use of the concept of "Euler Contribution". (See e.g. [7,8,10,2].) Let v be any vertex of a graph G minimally embedded on a surface Σ. Define the *Euler contribution* of vertex v to be

$$\Phi(v) = 1 - \frac{d(v)}{2} + \sum_{i=1}^{d(v)} \frac{1}{x_i}, \tag{1.1}$$

where the sum runs over the face angles at vertex v and x_i denotes the size of the ith face at v. (One should keep in mind here that a face may contribute more than one face angle at a vertex v. K_5 has such a minimal embedding on the torus, for example.)

Let $\chi(\Sigma)$ denote the *Euler characteristic* of the surface Σ; that is, $\chi(\Sigma) = 2 - 2g$ if Σ has orientable genus g and $\chi(\Sigma) = 2 - \overline{g}$ if Σ has non-orientable genus $\overline{g}$.

The next result is essentially due to Lebesgue [4].

Lemma 1.5. *If a connected graph G is minimally embedded on the surface Σ, then $\sum_{v \in V(G)} \Phi(v) = \chi(\Sigma)$.*

We define a vertex v to be a *control vertex* if $\Phi(v) \geq \chi(\Sigma)/V(G)$. Let $d(v)$ denote the degree of vertex v. Lemma 2.5 of [2] states

Lemma 1.6. *If G is minimally embedded in surface Σ, then for each control vertex v, if any,*

$$d(v) \leq 6 - 6\chi(\Sigma)/|V(G)|.$$