Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 308 (2008) 5937–5943
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Bipartite graphs are not universal fixers

R.G. Gibson[1]

Department of Mathematics, Simon Fraser University, 8888 University Drive, Burnaby, BC, Canada V5A 1S6

Received 20 October 2006; received in revised form 4 November 2007; accepted 5 November 2007
Available online 11 December 2007

Abstract

For any permutation π of the vertex set of a graph G, the graph πG is obtained from two copies G' and G'' of G by joining $u \in V(G')$ and $v \in V(G'')$ if and only if $v = \pi(u)$. Denote the domination number of G by $\gamma(G)$. For all permutations π of $V(G)$, $\gamma(G) \leq \gamma(\pi G) \leq 2\gamma(G)$. If $\gamma(\pi G) = \gamma(G)$ for all π, then G is called a universal fixer. We prove that graphs without 5-cycles are not universal fixers, from which it follows that bipartite graphs are not universal fixers.

Keywords: Domination; Prisms of graphs; Universal fixers; Bipartite graphs

1. Introduction

Informally, we consider the following problem, first posed by Diana (Weizhen) Gu [5] in 1999. Given an arbitrary graph G, form a new graph πG by joining the vertices of two disjoint copies of G by some matching. As shown in [1] the domination number $\gamma(\pi G)$ lies between $\gamma(G)$ and $2\gamma(G)$. Which are the graphs that always satisfy $\gamma(\pi G) = \gamma(G)$, regardless of the matching used to construct πG? More precisely, are there *any* graphs with nonempty edge sets for which this is true?

It was conjectured by Mynhardt and Xu [8] that such graphs did not exist. We prove this conjecture for a class of graphs that contains all bipartite graphs.

2. Definitions

Formally, for any permutation π of $V(G)$, the *prism of G with respect to π* is the graph πG obtained from two copies G' and G'' of G by joining $u \in V(G')$ and $v \in V(G'')$ if and only if $v = \pi(u)$. If π is the identity $\mathbf{1}_G$, then $\pi G = G \times K_2$, the *Cartesian product* of G and K_2. The graph $G \times K_2$ is often referred to as the *prism of* (or *over*) G and this serves as the motivation for our terminology above. Recent publications on prisms of graphs, sometimes also called permutation graphs, include [3,4].

For a graph G, we often write V and E for the vertex and edge sets of G respectively when it is unnecessary to emphasize the graph G. For $v \in V$, the *open neighbourhood* $N(v)$ of v is defined by $N(v) = \{u \in V : uv \in E\}$, and

E-mail address: richardg@sfu.ca.

[1] Recipient of an Undergraduate Student Research Award from the Canadian National Science and Engineering Research Council, Summer 2006.

0012-365X/$ - see front matter
doi:10.1016/j.disc.2007.11.006

the *closed neighbourhood* $N[v]$ of v is the set $N(v) \cup \{v\}$. For $S \subseteq V$, $N(S) = \bigcup_{s \in S} N(s)$ and $N[S] = \bigcup_{s \in S} N[s]$. For $A, B \subseteq V$, $N_A(B) = N(B) \cap A$; when $B = \{u\}$ we write $N_A(u)$ instead of $N_A(B)$. A set $S \subseteq V$ *dominates* G, written $S \succ G$, if every vertex in $V - S$ is adjacent to a vertex in S, i.e. if $V = N[S]$. The *domination number* $\gamma(G)$ of G is defined by $\gamma(G) = \min\{|S| : S \succ G\}$. A γ*-set* of G is a dominating set of G of cardinality $\gamma(G)$. A set $S \subseteq V$ is a 2*-packing* of G if $N[u] \cap N[v] = \phi$ (i.e. $d(u, v) \geq 3$) for all distinct $u, v \in S$. We follow [7] for domination terminology.

As shown in e.g. [6,8], $\gamma(G) \leq \gamma(\pi G) \leq 2\gamma(G)$ for all permutations π of $V(G)$. If $\gamma(\pi G) = \gamma(G)$ for some permutation π of $V(G)$, we call G a π*-fixer*. If G is a $\mathbf{1}_G$-fixer, that is, if $\gamma(G \times K_2) = \gamma(G)$, then G is a *prism fixer*, and if $\gamma(\pi G) = \gamma(G)$ for all permutations π of $V(G)$, then G is a *universal fixer*.

Prism fixers were also studied by Burger, Mynhardt and Weakley [1] and Hartnell and Rall [6], while universal fixers were first considered in [8], where it was conjectured that the edgeless graphs are the only universal fixers. (The graphs $\overline{K_n}$, $n \geq 1$, are universal fixers because $\pi \overline{K_n} = nK_2$ for all permutations π of $V(G)$, and $\gamma(\overline{K_n}) = \gamma(nK_2) = n$.)

Conjecture 1 (*[8]*). *If G is a nontrivial, connected graph, then G is not a universal fixer.*

The purpose of this paper is to prove that Conjecture 1 is true for bipartite graphs. We indeed prove a stronger result, namely that graphs without 5-cycles, induced or otherwise, satisfy Conjecture 1. This extends the work in [2], where it was proved that Conjecture 1 is true for regular bipartite graphs. Our work depends on the results in [6,8] which we state in Section 3.

3. Known results and more definitions

The first result characterizes π-fixers in terms of the existence of γ-sets with certain properties.

Lemma 1 (*[8]*). *Let G be a connected graph of order* $n \geq 2$ *and* π *a permutation of* V. *Then* $\gamma(\pi G) = \gamma(G)$ *if and only if G has a* γ*-set D such that*

(a) *D admits a partition* $D = D_1 \cup D_2$*, where* $D_1 \succ V - D_2$*;*
(b) $\pi(D)$ *is a* γ*-set of G and* $\pi(D_2) \succ V - \pi(D_1)$.

A γ-set D is called a *separable* γ*-set*, or a D_1*-*γ*-set* to emphasize the set D_1, if D satisfies Lemma 1(a). Further properties of such sets were obtained in [8]. For $X, Y \subseteq V$ we denote the set of all edges joining vertices in X to vertices in Y by $E(X, Y)$.

Lemma 2 (*[8]*). *Suppose* $D = D_1 \cup D_2$ *is a* D_1*-*γ*-set of G. Then*

(a) D_2 *is a 2-packing of G;*
(b) $E(D_1, D_2) = \phi$*;*
(c) $\sum_{x \in D_1} \deg x \geq n - \gamma$, $\sum_{x \in D_2} \deg x \leq n - \gamma$.

For a D_1-γ-set, $D = D_1 \cup D_2$ of G and a permutation π of V, if $\pi(D)$ is a $\pi(D_2)$-γ-set of G (thus if Lemma 1(b) holds), we say that D is *effective under* π; otherwise D is *ineffective under* π. The following theorem follows from Lemma 1.

Theorem 3 (*[8]*). *The nontrivial graph G is a universal fixer if and only if for each permutation* π *of V there exists a separable* γ*-set of G that is effective under* π.

By Lemma 2(c), if D is a D_1-γ-set, then $\sum_{x \in D_2} \deg x \leq n - |D|$. If $\sum_{x \in D_2} \deg x = n - |D|$, then Lemma 2(a) and (b) imply that D_2 dominates $V - D$ and thus D is also a D_2-γ-set. We then call D a *symmetric* γ-set and assume implicitly that D admits a partition $D = D_1 \cup D_2$ such that D is a D_i-γ-set for $i = 1, 2$; otherwise D is called an *asymmetric* γ-set. We henceforth denote symmetric γ-sets by A (usually) or B or C, and asymmetric γ-sets by D. Symmetric γ-sets were also studied in [6], where they were called *two-colored* γ-sets. The following lemma and theorem give more information on symmetric γ-sets and prism fixers.

Lemma 4 (*[6,8]*). *Suppose that A is a symmetric* γ*-set of G. Then*

(a) *A is independent;*
(b) $\delta(G) \geq 2$*;*