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H-triangles with k interior H-points ☆

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Abstract

An H-triangle is a triangle with corners in the set of vertices of a tiling of $\mathbb{R}^2$ by regular hexagons of unit edge. It is known that any H-triangle with exactly 1 interior H-point can have at most 10 H-points on its boundary. In this note we prove that any H-triangle with exactly k interior H-points can have at most $3k+7$ boundary H-points. Moreover we form two conjectures dealing with H-polygons.

Keywords: Hexagonal tile; H-point; H-triangle; Triangular lattice; Pick's theorem

1. Introduction

Let H be the set of vertices of a tiling of $\mathbb{R}^2$ by regular hexagons with unit edge. A point of H is called an H-*point*, a simple polygon in $\mathbb{R}^2$ whose corners lie in H is called an H-*polygon*. For a planar H-polygon P we denote $b(P) = |H \bigcap \partial P|$ and $i(P) = |H \bigcap \operatorname{int} P|$. H can be considered as the union of two disjoint triangular lattices denoted by H^+, H^- such that for any two points in H^+ (H^-) there exists a translation of the plane which maps one of the two points to the other and H to H. A point of H^+ (H^-) is called an H^+-*point* (H^--*point*). Two points x and y are said to be *equivalent* if $x, y \in H^+$ or $x, y \in H^-$. Otherwise we say that x, y are *non-equivalent*.

Let A denote the set of all centers of the hexagonal tiles which determine H. A point of A is called an A-*point*. Clearly, $H^+ \bigcup H^- \bigcup A$ forms a *triangular lattice* with the area of each triangular tile $\frac{\sqrt{3}}{4}$. We will denote this triangular lattice by $T = H^+ \bigcup H^- \bigcup A$, and a point of T is called a T-*point*. A simple polygon in $\mathbb{R}^2$ with all corners in T (A) is called a T-*polygon* (an A-*polygon*). Clearly an H-polygon or an A-polygon is also a T-polygon. For the related research see [7,2,5] and [1].

It is known that for an H-triangle $\triangle$ with exactly one interior H-point, $b(\triangle) \in \{3, 4, 5, 6, 7, 8, 10\}$ (see [3]). On the basis of [3] this paper extends the result to H-triangles with exactly k interior H-points, showing that any H-triangle with exactly k interior H-points can have at most $3k + 7H$-points on its boundary.

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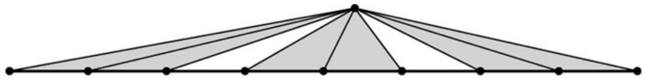

Fig. 1. A-triangles consisting of $2k$ smaller A-triangles contain at least $\lfloor \frac{2k+1}{3} \rfloor \times 2 \geq k+1$ H-points.

2. Basic facts and related lemmas

A segment with endpoints in T (H, A) is called a T-*segment* (an H-*segment*, an A-*segment*).

Lemma 1 ([3]). *Assume that the origin of $\mathbb{R}^2$ lies in H^+. If $x \in A(H^-)$, then $2x \in H^-(A)$ and $3x \in H^+$.*

Lemma 2 ([3]). *If xy is an H-segment such that $xy \bigcap H = \{x, y\}$ and $xy \bigcap A = \{a\}$, then the point a is the midpoint of xy. Moreover, x and y are non-equivalent.*

Lemma 3 ([3]). *If an H-segment contains two A-points, then between them there are at least two H-points which are non-equivalent.*

From Lemma 3 immediately we have the following lemma.

Lemma 4. *Any A-segment contains zero or an even number of H-points.*

Lemma 5. *Let $\triangle$ be an A-triangle in which every side contains exactly 2 A-points. If there exists one side of $\triangle$ which contains H-points, then $\triangle$ contains at least 2 H-points, otherwise $\triangle$ contains at least 1 H-point in its interior.*

Proof. If one side of $\triangle$ contains H-points, then, by Lemma 3, it must contain 2 H-points, and so $\triangle$ contains at least 2 H-points. If no side of $\triangle$ contains an H-point, by an argument as in the proof of Theorem 7 of [4], we can conclude that $\triangle$ contains at least 1 H-point in its interior. □

As in [3] we will use the notion of *level* of a T-triangle $\triangle = \triangle xyz$, here the corners x, y, z of $\triangle$ are labeled in such a way that xy has the largest number of points from $T = H \bigcup A$. The line containing xy is denoted by l_0. From l_0 to z draw all lines $l_1, l_2, \ldots, l_s$ passing through T-points that are parallel to l_0 and intersect $\triangle$. Clearly $z \in l_s$, and the distance between l_j and l_{j+1} is the same for every j. We say that $\triangle$ has s levels $l_1, l_2, \ldots, l_s$. It is obvious that every H-triangle with one interior H-point has at least two levels.

Let m_j be the *relative length* of $\triangle \bigcap l_j$, that is, the length with the unit length being the distance between two consecutive T-points on l_j. Denote $t_j = |l_j \bigcap \operatorname{int} \triangle \bigcap T|$. Obviously for $j > 0$ we have $\lfloor m_j \rfloor - 1 \leq t_j \leq \lfloor m_j \rfloor + 1$, where $\lfloor . \rfloor$ denotes the greatest integer function. Notice that if $t_j = \lfloor m_j \rfloor - 1$, then m_j is an integer, and $m_j = t_j + 1$.

Lemma 6 ([3]). *If an H-triangle $\triangle$ has s levels, then for $0 \leq j < s$ we have $m_j = m_0(1 - \frac{j}{s})$.*

Lemma 7. *There exists no H-triangle $\triangle$ with k interior H-points and $2k+2$ interior A-points.*

Proof. For $k = 1$ the conclusion is correct (see [3]). Now consider $k \geq 2$.

Let $\triangle$ be an H-triangle with exactly k interior H-points. Assume on the contrary that $\triangle$ contains $2k+2$ interior A-points.

Case 1: The $2k+2$ interior A-points of $\triangle$ are non-collinear.

Denote by P the point set consisting of the $2k+2$ A-points not on a line.

Case 1.1: There are $2k+1$ A-points of P on a line, say l.

Then $CH(P)$, the convex hull of P, is an A-triangle consisting of $2k$ smaller A-triangles, and the interiors of these A-triangles do not intersect. By Lemma 5, we know that any two smaller A-triangles with a common side contain at least two H-points. So, $CH(P)$ contains at least $\lfloor \frac{2k+1}{3} \rfloor \times 2 \geq k+1$ H-points (see Fig. 1), which implies that $\operatorname{int} \triangle$ contains at least $k+1$ H-points, a contradiction.

Case 1.2: There are at most $2k$ A-points of P collinear.

Now deduce by induction on k a contradiction that $\operatorname{int} \triangle$ contains at least $k+1$ H-points.

When $k = 2$, there are at most $2k = 4$ A-points of P collinear, and we can obtain at least three A-triangles in which every side contains exactly 2 A-points, and the interiors of these three A-triangles do not intersect. By Lemma 5 we