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Kernels and some operations in edge-coloured digraphs

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Abstract

Let D be an edge-coloured digraph, $V(D)$ will denote the set of vertices of D; a set $N \subseteq V(D)$ is said to be a kernel by monochromatic paths of D if it satisfies the following two conditions: For every pair of different vertices $u, v \in N$ there is no monochromatic directed path between them and; for every vertex $x \in V(D) - N$ there is a vertex $y \in N$ such that there is an xy-monochromatic directed path.

In this paper we consider some operations on edge-coloured digraphs, and some sufficient conditions for the existence or uniqueness of kernels by monochromatic paths of edge-coloured digraphs formed by these operations from another edge-coloured digraphs.

Keywords: Kernel; Kernel by monochromatic paths; m-coloured digraph

1. Introduction

For general concepts we refer the reader to [1]. In the paper we write digraph to mean 1-digraph in the sense of Berge [1]. In this paper D will denote a possibly infinite digraph; $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D, respectively. If S is a nonempty subset of $V(D)$ then the subdigraph $D[S]$ induced by S is the digraph with vertex set S and whose arcs are those arcs of D which join vertices of S.

A directed path is a finite or infinite sequence $(x_1, x_2, \ldots)$ of distinct vertices of D such that $(x_i, x_{i+1}) \in A(D)$ for each i. When D is infinite and the sequence is infinite we call the directed path an infinite outward path. If T is a directed path and $a, b, \in V(T)$, (a, T, b) will denote the ab-directed path contained in T. (When a appears before b in T).

A set $I \subseteq V(D)$ is independent if $A(D[I]) = \emptyset$. A kernel N of D is an independent set of vertices such that for each $z \in V(D) - N$ there exists a zN-arc in D.

We call the digraph D an m-coloured digraph if the arcs of D are coloured with m colours. A directed path or a directed cycle is called monochromatic if all of its arcs are coloured a like.

If D is an m-coloured digraph then the closure of D, denoted $\mathcal{C}(D)$ is the digraph defined as follows: $V(\mathcal{C}(D)) = V(D)$ $A(\mathcal{C}(D)) = \{(u, v) \mid \text{there exists an } uv\text{-monochromatic directed path contained in } D\}$.

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Notice that for any m-coloured digraph D; N is a kernel by monochromatic paths of D iff N is a kernel of $\mathcal{C}(D)$. (Although the concept of kernel was defined in [1] for 1-digraphs, the same concept is valid and can be considered in multidigraphs.)

The concept of kernel was introduced by Von Neumann and Morgenstern [18] in the context of Game Theory. The problem of the existence of a kernel in a given digraph has been studied by several authors in particular by Richardson [14,15], Duchet and Meynel [5], Duchet [3,4], Galeana-Sánchez and Neumann-Lara [9].

The existence of kernels of digraphs formed by some operations from another digraphs have been studied by several authors, namely, M.Blidia, P. Duchet, H. Jacob, F. Maffray and H. Meyniel [2], Jerzy Topp [17], Galeana-Sánchez [6], Galeana-Sánchez and Neumann-Lara [10,11]. The concept of kernel by monochromatic paths generalizes the concept of kernel of a digraph and has been studied by several authors: Sauer, Sands and Woodrow [16], Shen Minggang [13], Galeana-Sánchez [7,8].

In [17] Jerzy Topp defined the digraphs $S(D)$, $Q(D)$, $T(D)$ and $L(D)$ which were called the subdivision digraph, the middle digraph, the total digraph and the line digraph of D respectively; and studied some necessary or sufficient conditions for the existence or uniqueness of kernels of these digraphs.

In this paper we define the following digraphs: the subdivision $S(D)$, a generalization of the subdivision $S'(D)$, the digraph $R'(D)$, the middle digraph $Q(D)$ and the total digraph $T(D)$, for an m-coloured digraph D. Also it is proved the following results: If D has no monochromatic infinite outward path, then $S(D)$ (resp. $S'(D)$ and $R'(D)$) has a kernel by monochromatic paths.

The number of kernels by monochromatic paths of D is less than or equal to the number of kernels by monochromatic paths of $Q(D)$(resp. $T(D)$).

If D has no monochromatic directed cycle then the number of kernels by monochromatic paths of D is equal to the number of kernels by monochromatic paths of $Q(D)$ (resp. $T(D)$).

2. Kernels by monochromatic paths in the subdivision digraph of an m-coloured digraph

In [17] was proved that the subdivision digraph of any digraph has a kernel, in this section we define the subdivision digraph $S(D)$ of an m-coloured digraph D and it is proved that if D has no monochromatic infinite outward path, then D has a kernel by monochromatic paths.

Let $D = (V(D), A(D))$ be an m-coloured digraph, we define the functions Γ_D, $\Gamma_{D,i}$, Γ_D^-, $\Gamma_{D,i}^-$, from $V(D)$ to $\mathcal{P}(V(D))$ as follows. For any $u \in V(D)$;

$$\Gamma_D(u) = \{v \in V(D) \mid (u, v) \in A(D)\}, \qquad \Gamma_D^-(u) = \{v \in V(D) \mid (v, u) \in A(D)\},$$
$$\Gamma_{D,i}(u) = \{v \in V(D) \mid (u, v) \in A(D) \text{ and } (u, v) \text{ is } i\text{-coloured}\},$$
$$\Gamma_{D,i}^-(u) = \{v \in V(D) \mid (v, u) \in A(D) \text{ and } (v, u) \text{ is } i\text{-coloured}\}.$$

If $U \subseteq V(D)$, we denote $\Gamma_D(U) = \bigcup_{u \in U} \Gamma_D(u)$.

Definition 2.1. Let D be an m-coloured digraph, we define the subdivision digraph $S(D)$ of D as follows:

$$V(S(D)) = V(D) \cup A(D) \quad \text{and}$$
$$\Gamma_{S(D),i}(x) = \begin{cases} \{x\} \times \Gamma_{D,i}(x) & \text{if } x \in V(D), \\ \{v\} & \text{if } x = (u, v) \in A(D) \text{ and } v \in \Gamma_{D,i}(u). \end{cases}$$

Notice that for a vertex x of the subdivision digraph we have the following: If x corresponds to a vertex of D then x is adjacent toward the arcs which incide from x in D, preserving the colour of those arcs; and if x corresponds to an arc of D then x is adjacent only toward the terminal endpoint of x preserving the colour of x. Also notice that $S(D)$ is obtained from D by changing each arc of D for a directed path of length two with the same colour as the arc.

Lemma 2.1. *Let D be an m-coloured digraph, $S(D)$ its subdivision digraph and $a, b, c \in V(S(D))$ such that $b \in A(D)$, $a \neq b$ and $b \neq c$. If T_1, is an ab-monochromatic directed path in $S(D)$ and T_2 is a bc-monochromatic directed path in $S(D)$, then T_1, and T_2 are coloured alike.*

Proof. We may assume $b = (u, v) \in A(D)$. Clearly $\Gamma_{S(D)}^-(b) = \{u\}$ and $\Gamma_{S(D)} = \{v\}$, since $a \neq b$ and $b \neq c$ we have $\ell(T_i) \geq 1$, $i \in \{1, 2\}$ and thus $(u, b) \in A(T_1)$ and $(b, v) \in A(T_2)$, so from the definition of $S(D)$, (u, b) and (b, v) are coloured alike; so T_1 and T_2 are coloured alike. □