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ScienceDirect
Discrete Mathematics 308 (2008) 6064–6077
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Contractible subgraphs, Thomassen's conjecture and the dominating cycle conjecture for snarks

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Received 6 July 2004; received in revised form 13 November 2007; accepted 14 November 2007
Available online 26 December 2007

Abstract

We show that the conjectures by Matthews and Sumner (every 4-connected claw-free graph is Hamiltonian), by Thomassen (every 4-connected line graph is Hamiltonian) and by Fleischner (every cyclically 4-edge-connected cubic graph has either a 3-edge-coloring or a dominating cycle), which are known to be equivalent, are equivalent to the statement that every snark (i.e. a cyclically 4-edge-connected cubic graph of girth at least five that is not 3-edge-colorable) has a dominating cycle.

We use a refinement of the contractibility technique which was introduced by Ryjáček and Schelp in 2003 as a common generalization and strengthening of the reduction techniques by Catlin and Veldman and of the closure concept introduced by Ryjáček in 1997.

Keywords: Dominating cycle; Contractible graph; Cubic graph; Snark; Line graph; Hamiltonian graph

1. Introduction

In this paper we consider finite undirected graphs. All the graphs we consider are loopless (with one exception in Section 3); however, we allow the graphs to have multiple edges. We follow the most common graph-theoretic terminology and notation, and for concepts and notation not defined here we refer the reader to [2]. If F, G are graphs then $G - F$ denotes the graph $G - V(F)$ and by an a, b*-path* we mean a path with end vertices a, b. A graph G is *claw-free* if G does not contain an induced subgraph isomorphic to the claw $K_{1,3}$.

In 1984, Matthews and Sumner [8] posed the following conjecture.

Conjecture A (*[8]*). *Every 4-connected claw-free graph is Hamiltonian.*

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0012-365X/$ - see front matter
doi:10.1016/j.disc.2007.11.026

Since every line graph is claw-free (see [1]), the following conjecture by Thomassen is a special case of Conjecture A.

Conjecture B ([12]). *Every 4-connected line graph is Hamiltonian.*

A closed trail T in a graph G is said to be *dominating*, if every edge of G has at least one vertex on T, i.e., the graph $G - T$ is edgeless (a closed trail is defined as usual, except that we allow a single vertex to be such a trail). The following result by Harary and Nash-Williams [6] shows the relation between the existence of a dominating closed trail (abbreviated DCT) in a graph G and Hamiltonicity of its line graph $L(G)$.

Theorem 1 ([6]). *Let G be a graph with at least three edges. Then $L(G)$ is Hamiltonian if and only if G contains a DCT.*

Let k be an integer and let G be a graph with $|E(G)| > k$. The graph G is said to be *essentially k-edge-connected* if G contains no edge cut R such that $|R| < k$ and at least two components of $G - R$ are nontrivial (i.e. containing at least one edge). If G contains no edge cut R such that $|R| < k$ and at least two components of $G - R$ contain a cycle, G is said to be *cyclically k-edge-connected*.

It is well-known that G is essentially k-edge-connected if and only if its line graph $L(G)$ is k-connected. Thus, the following statement is an equivalent formulation of Conjecture B.

Conjecture C. *Every essentially 4-edge-connected graph contains a DCT.*

By a *cubic* graph we will always mean a regular graph of degree 3 without multiple edges. It is easy to observe that if G is cubic, then a DCT in G becomes a dominating cycle (abbreviated DC), and that every essentially 4-edge-connected cubic graph must be triangle-free, with a single exception of the graph K_4. To avoid this exceptional case, we will always consider only essentially 4-edge-connected cubic graphs on at least five vertices.

Since a cubic graph is essentially 4-edge-connected if and only if it is cyclically 4-edge-connected (see [5], Corollary 1), the following statement, known as the Dominating Cycle Conjecture, is a special case of Conjecture C.

Conjecture D. *Every cyclically 4-edge-connected cubic graph has a DC.*

Restricting to cyclically 4-edge-connected cubic graphs that are not 3-edge-colorable, we obtain the following conjecture posed by Fleischner [4].

Conjecture E ([4]). *Every cyclically 4-edge-connected cubic graph that is not 3-edge-colorable has a DC.*

In [10], a closure technique was used to prove that Conjectures A and B are equivalent. Fleischner and Jackson [5] showed that Conjectures B–D are equivalent. Finally, Kochol [7] established the equivalence of these conjectures with Conjecture E. Thus, we have the following result.

Theorem 2 ([5,7,10]). *Conjectures A–E are equivalent.*

A cyclically 4-edge-connected cubic graph G of girth $g(G) \geq 5$ that is not 3-edge-colorable is called a *snark*. Snarks have turned out to be an important class of graphs, for example in the context of nowhere zero flows. For more information about snarks see the paper [9]. Restricting our considerations to snarks, we obtain the following special case of Conjecture E.

Conjecture F. *Every snark has a DC.*

The following theorem, which is the main result of this paper, shows that Conjecture F is equivalent to the previous ones.

Theorem 3. *Conjecture F is equivalent to Conjectures A–E.*

The proof of Theorem 3 is postponed to Section 4.

As already noted, every cyclically 4-edge-connected cubic graph other than K_4 must be triangle-free. Thus, the difference between Conjectures E and F consists in restricting to graphs which do not contain a 4-cycle. For the proof