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The chromatic number of 5-valent circulants

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Abstract

A circulant $C(n; S)$ with connection set $S = \{a_1, a_2, \dots, a_m\}$ is the graph with vertex set $\mathbb{Z}_n$, the cyclic group of order n, and edge set $E = \{\{i, j\} : |i - j| \in S\}$. The chromatic number of connected circulants of degree at most four has been previously determined completely by Heuberger [C. Heuberger, On planarity and colorability of circulant graphs, Discrete Math. 268 (2003) 153–169]. In this paper, we determine completely the chromatic number of connected circulants $C(n; a, b, n/2)$ of degree 5. The methods used are essentially extensions of Heuberger's method but the formulae developed are much more complex.

Keywords: Chromatic number; Circulant

1. Introduction

Let n and $1 \le a_1 < a_2 < \cdots < a_m \le \lfloor n/2 \rfloor$ be positive integers. A graph $G = (V, E)$ is called a *circulant* of order n if $V = \{0, 1, \dots, n-1\}$, $E = \{\{i, i + a_j \pmod n\} : 0 \le i \le n-1, 1 \le j \le m\}$, and is denoted by $C(n; a_1, a_2, \dots, a_m)$.

The chromatic number of circulants of degree ≤ 4 has been completely determined in [2]. In this paper we deal with connected 5-valent circulants. Therefore $n \ge 6$ is even, $m = 3$ and $a_3 = n/2$. We use the notation $G = C(n; a, b, n/2)$, where $1 \le a, b < n/2$ and $a \ne b$. Notice that G is connected iff $\gcd(a, b, n/2) = 1$.

The following result completely determines the chromatic number of connected 5-valent circulants.

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Theorem 1. *Let $n \geq 6$ be even and $G = C(n; a, b, n/2)$ be a connected circulant. Let $g = \gcd(a, b, n)$. Then*

$$\chi(G) = \begin{cases} 6 & \text{if } G = C(6; 1, 2, 3) \\ 5 & \text{if } G = C(10; 2, 4, 5) \\ 5 & \text{if } 4 \nmid n \text{ and } a + b = n/2 \\ 4 & \text{if } 4 \mid n \text{ and } a + b = n/2 \\ 4 & \text{if } b = n/4 \text{ or } a = n/4 \\ 4 & \text{if } (g = 1 \text{ or } 6 \nmid n), (b \equiv \pm 2a \pmod{n} \text{ or } a \equiv \pm 2b \pmod{n}), \\ & G \neq C(10; 2, 4, 5) \text{ and } G \neq C(6; 1, 2, 3) \\ 4 & \text{if } n = 16, 24, 28 \text{ and } (b \equiv \pm 5a \pmod{n} \text{ or } a \equiv \pm 5b \pmod{n}) \\ 4 & \text{if } n = 26, g = 2 \text{ and } (b \equiv \pm 5a \pmod{n} \text{ or } a \equiv \pm 5b \pmod{n}) \\ 4 & \text{if } n = 20 \text{ and } (b \equiv \pm 6a \pmod{n} \text{ or } a \equiv \pm 6b \pmod{n}) \\ 4 & \text{if } n = 22, g = 1 \text{ and } (b \equiv \pm 8a \pmod{n} \text{ or } a \equiv \pm 8b \pmod{n}) \\ 4 & \text{if } n = 28 \text{ and } (b \equiv \pm 8a \pmod{n} \text{ or } a \equiv \pm 8b \pmod{n}) \\ 4 & \text{if } n = 40 \text{ and } (b \equiv \pm 11a \pmod{n} \text{ or } a \equiv \pm 11b \pmod{n}) \\ 2 & \text{if } 4 \nmid n \text{ and both } a, b \text{ are odd} \\ 3 & \text{otherwise.} \end{cases}$$

2. Preliminaries

Given a graph $G = (V, E)$ and a group $H \leq \operatorname{Aut} G$ we define a quotient graph $\bar{G} = G/H = (\bar{V}, \bar{E})$ by taking $\bar{V}$ to be the orbits of the action of H on V and joining orbits $\bar{v}$ to $\bar{u}$ if and only if there exist adjacent vertices $v \in \bar{v}$ and $u \in \bar{u}$. If every orbit is formed by an independent set of vertices then the mapping $u \mapsto \bar{u}$ defines a graph homomorphism $G \to \bar{G}$ and clearly a k-coloring of $\bar{G}$ lifts to a k-coloring of G.

Notice that, since $G = C(n; a, b, n/2)$ is connected, integers a, b and $n/2$ generate the cyclic group of order n. Moreover, $g = \gcd(a, b, n) = 1$ or $g = 2$. Thus a subgraph $C(n; a, b)$ of G is either connected or it has exactly two connectivity components. We say that two circulants $G = C(n; a, b, \frac{n}{2})$ and $\bar{G} = C(n; \bar{a}, \bar{b}, \frac{n}{2})$ of order n are *multiplier-isomorphic* if there is an isomorphism of the form $V(G) \ni i \mapsto mi \in V(\bar{G})$ for some multiplier $m \in \mathbb{Z}_n$. This implies the following claim.

Claim 2. *A connected 5-valent circulant $C(n; a, b, \frac{n}{2})$ is multiplier-isomorphic with a circulant $C(n; \bar{a}, \bar{b}, \frac{n}{2})$, where either $\gcd(\bar{a}, \bar{b}) = 1$ or $\gcd(\bar{a}, \bar{b}) = 2$; in the latter case $n \equiv 2 \bmod 4$.* □

Moreover, if $\gcd(a, n) = 1$ (or $\gcd(b, n) = 1$) we can easily transform G to a circulant of the form $C(n; 1, \bar{b}, \frac{n}{2})$.

Claim 3. $\gcd(a, n) = 1$ *then a circulant $C(n; a, b, \frac{n}{2})$ is multiplier-isomorphic with a circulant $C(n; 1, \bar{b}, \frac{n}{2})$, where $\bar{b} = a^{-1}b \pmod{n}$.* □

From now on we assume that G is not K_6. By Brooks' theorem [1], we get an upper bound.

Claim 4. $\chi(G) \leq 5$. □

If a, b and $\frac{n}{2}$ are odd integers then $G = C(n; a, b, \frac{n}{2})$ is a bipartite graph and immediately we get the following.

Lemma 5. *Let $n \equiv 2 \pmod 4$ and a, b be odd. Then $G = C(n; a, b, \frac{n}{2})$ is 2-colorable.* □

3. Case $a = 1$

Lemma 6. *Let $n \equiv 2 \pmod 4$, b is even and $4 \leq b < \frac{n}{4}$. Then $G = C(n; 1, b, \frac{n}{2})$ is 3-colorable.*

Proof. Let $n = 2m$. Then m is odd. Denote by x and r unique integers satisfying the equality $m = bx + r$, where $0 \leq r < b$. Hence $x \geq 2$, r is odd and $1 \leq r \leq b - 1$. Given x and r, we define a 3-coloring of G by specifying a coloring sequence $w_{x,r}$ of length n over the alphabet $\{B, G, R\}$.

I. x is even. Let