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Combinatorial properties of a general domination problem with parity constraints

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Abstract

We consider various properties of a general parity domination problem: given a graph G on n vertices, one is looking for a subset S of the vertex set such that the open/closed neighborhood of each vertex contains an even/odd number of vertices in S (it is prescribed individually for each vertex which of these applies). We define the parameter $s(G)$ to be the number of solvable instances out of 4^n possibilities and study the properties of this parameter. Upper and lower bounds for general graphs and trees are given as well as a remarkable recurrence formula for rooted trees. Furthermore, we give explicit formulas in several special cases and investigate random graphs.

Keywords: Domination problem; Parity constraint; Matrix algebra

1. Introduction

The classical problem of domination asks for a subset S of the vertex set of a graph $G = (V(G), E(G))$ of minimum cardinality such that $N[v] \cap S \neq \emptyset$ for all $v \in V(G)$, where $N[v] = \{u \in V(G) \mid \exists e = (u, v) \in E(G)\} \cup \{v\}$ denotes the closed neighborhood of v. Quite a lot of different modifications and generalizations of this problem are known. For instance, the *k-tuple domination problem* [16] asks for a minimum set S such that $|N[v] \cap S| \geq k$ for all vertices v. Similarly, in the *k-domination problem* [9,10] the task is to find a set S of minimum cardinality such that $|N(v) \cap S| \geq k$ for all vertices v, where $N(v) = \{u \in V(G) \mid \exists e = (u, v) \in E(G)\}$ denotes the open neighborhood of v. Even more generally, one can prescribe a set R_v for every vertex v and pose the question whether there exists a set S such that $|N[v] \cap S| \in R_v$ (or $|N(v) \cap S| \in R_v$) for all vertices v.

The special cases $R_v = \{1, 2, 3, \ldots\}$ and $R_v = \{k, k + 1, \ldots\}$ have already been mentioned. These and other variants, such as $R_v = \{1\}$, are discussed in the book of Haynes, Hedetniemi and Slater [17]. Another interesting kind of domination problem involves parity constraints. It has been treated in a series of papers [1–4,8], motivated by the following remarkable result of Sutner [19]:

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Theorem 1 (*Sutner [19]*). *For every graph $G = (V(G), E(G))$, there exists a set $S \subseteq V(G)$ such that $|N[v] \cap S|$ is odd for every $v \in V(G)$.*

This means that the domination problem for $R_v = \{1, 3, 5, \ldots\}$ is always solvable if we consider closed neighborhoods. Thus, it is natural to consider a general parity assignment problem, where each R_v is either $\{1, 3, 5, \ldots\}$ or $\{0, 2, 4, \ldots\}$. It has been treated quite extensively in [1–4], where the notions of "parity dimension" and "all parity realizable graphs" have been introduced. Wagner [21] gives a recursive procedure for determining the parity dimension of a tree, which is then applied to enumeration problems involving the parity dimension, in particular to counting all parity realizable trees. All these papers deal with parity domination with respect to closed neighborhoods, but analogous results exist for the case of open neighborhoods as well—see for instance [14] and the references therein. Other works such as [15,20] study parity domination with a focus on complexity results.

In another recent paper, Gassner and Hatzl [13] discuss an even more general parity domination problem from an algorithmic point of view: for every vertex v, we impose exactly one of the following four constraints:

- $|N(v) \cap S| \equiv 0 \bmod 2$,
- $|N(v) \cap S| \equiv 1 \bmod 2$,
- $|N[v] \cap S| \equiv 0 \bmod 2$,
- $|N[v] \cap S| \equiv 1 \bmod 2$,

i.e., the open/closed neighborhood has to contain an even/odd number of vertices in S. One of the reasons to consider domination problems with parity constraints lies in the fact that the problem can be stated easily in terms of matrix algebra: in the following, we denote by A and $A + I$ the open neighborhood matrix (adjacency matrix) and the closed neighborhood matrix respectively (I is the identity matrix). Furthermore, we use a vector $a \in \{0, 1\}^{V(G)}$ as a representation for the neighborhood information (i.e., whether the open or closed neighborhood is considered for a certain vertex): If the entry a_v that corresponds to a vertex v is 0, the open neighborhood is of interest for this vertex, and the closed neighborhood otherwise. Another vector $b \in \{0, 1\}^{V(G)}$ represents the prescribed parities. Using these vectors, our requirements can be written as

$$(A + \operatorname{diag}(a))x = b \tag{1}$$

over the field $\mathbb{F}_2$. Obviously, $x_v = 1$ if and only if $v \in S$.

In this paper, we are interested in the number of solvable instances—a parameter that plays an analogous role to the parity dimension (for the domination problem with parity constraints considering closed subsets only): let the *solvability number* $s(G)$ denote the number of solvable instances for a graph G, i.e., the number of pairs $(a, b) \in \{0, 1\}^{V(G)} \times \{0, 1\}^{V(G)}$ such that there exists a vector x satisfying the system of linear equations in (1).

Basic linear algebra gives us the following simple lemma:

Lemma 2. *Let $G = (V(G), E(G))$ be a graph, then*

$$s(G) = \sum_{a \in \{0,1\}^{V(G)}} 2^{\operatorname{rk}(A+\operatorname{diag}(a))}, \tag{2}$$

where $\operatorname{rk}(B)$ *denotes the rank of a matrix B over* $\mathbb{F}_2$.

Remark 3. Replacing 2 by a variable x in the above formula, we obtain a polynomial

$$S_G(x) = \sum_{a \in \{0,1\}^{V(G)}} x^{\operatorname{rk}(A+\operatorname{diag}(a))}$$

with interesting properties: $S_G(0) = 1$ if G is the empty graph, and $S_G(0) = 0$ otherwise; $S_G(1) = 2^{|V(G)|}$, and $\frac{S'_G(1)}{S_G(1)}$ gives the average rank of $A + \operatorname{diag}(a)$, as a varies over all possible vectors.

Corollary 4.

$$2^{|V(G)|} \leq s(G) \leq 4^{|V(G)|}.$$