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Construction of some countable 1-arc-transitive bipartite graphs

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Abstract

We generalize earlier work which gave a method of construction for bipartite graphs which are obtained as the set of maximal or minimal elements of a certain cycle-free partial order. The method is extended here to produce a 1-arc-transitive bipartite graph in a 'free' way, starting with any partial order with greatest and least element and with instructions on its points about how they will ramify in the extension. A key feature of our work is the interplay between properties of the initial partial order, the extended partial order, and the bipartite graph which results. We also extend the earlier work by giving a complete characterization of all 2-CS-transitive cycle-free partial orders. In addition, we discuss the completeness of the constructed partial orders, in the sense of Dedekind and MacNeille, and remark that the bipartite graph constructed can only be 2-arc-transitive in the cycle-free case.

Keywords: Partially ordered sets; Cycle-free partial orders; Infinite graphs; Arc-transitive graphs

1. Introduction and preliminaries

The notion of a *cycle-free* partial order (*CFPO*) was originally proposed by Rubin in [9] and later redefined slightly and extensively developed by Warren in [14]. Since then a number of other papers on the subject have appeared [3,6, 11–13]. It was observed in [14] that a large class of interesting *CFPO*s are in fact two-level partial orders and so can be thought of in a natural way as bipartite graphs. See Theorem 2 below.

Let $M = (M, \leq)$ be a partially ordered set, also called a *poset*. We write $x \parallel y$ to mean that x and y are incomparable. Given a subset X of M we let $\bigvee^M X = \{a \in M : a \geq X\}$ and $\bigwedge^M X = \{a \in M : a \leq X\}$, where $a \geq X$ means that $(\forall x \in X)\ a \geq x$. A subset I of M is a *Dedekind ideal* (or just ideal) of M, written $I \in I^D(M)$, if $I \neq \varnothing$, $\bigvee^M I \neq \varnothing$, and $\bigwedge^M \bigvee^M I = I$. Given any $m \in M$ we let $\mathrm{PI}(m) = \{x \in M : x \leq m\}$, noting that $\mathrm{PI}(m)$ is an ideal of M. We call $\mathrm{PI}(m)$ the *principal ideal generated by* m. We say an ideal I is *principal* if $I = \mathrm{PI}(m)$ for some $m \in M$. Dually we use $\mathrm{PF}(m)$ to denote the *principal filter generated by* m. A poset is then said to be Dedekind–MacNeille *complete* (D–M complete) if every ideal is principal. A Dedekind–MacNeille complete poset has the property that whenever two elements have a lower bound, they have an infimum, and whenever they have an upper bound they have a supremum.

The Dedekind–MacNeille completion of M, written M^D, is defined to be the partial order with domain $I^D(M)$ ordered by inclusion. The partial order M can be embedded into M^D in a natural way (mapping elements to the

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principal ideals that they generate). It can be shown that the poset M^D is the least extension of M which is D–M complete.

Definition 1. Let M be a partial order, and let $a, b \in M$. Let $C = (c_0, c_1, \dots, c_n)$ be a sequence of points of M such that $c_0 = a$, $c_n = b$ and c_i is comparable with c_{i+1} for each $i < n$. Let σ_k (with $0 \le k < n$) be maximal chains in M^D with endpoints $c_k, c_{k+1} \in \sigma_k$ such that if $x \in \sigma_i \cap \sigma_j$ for some $i < j$, then $j = i + 1$ and $x = c_{i+1}$. Then we say that $\bigcup_{k<n} \sigma_k$ is a *path* from a to b in M.

The poset M is said to be *connected* if between any two points of M there is, in M^D, at least one path, and it is *cycle-free* if between any two points of M there is, in M^D, a unique path. The poset M is *k-CS-transitive* if for any two isomorphic connected substructures of M of size k there is an automorphism of M taking the first to the second, and *k-CS-homogeneous* if any isomorphism between two connected substructures of M of size k extends to an automorphism.

The full classification of k-CS-transitive *CFPO*s (for $k \ge 3$) is essentially complete. The only place where the classification is still not completely explicit is for the case that a certain poset ALT does not embed and $k \ge 5$; see [12]. For a reasonably detailed summary of the classification we refer the reader to [6, Section 3].

The connection between *CFPO*s and graphs is given by the following result, which we refer to as the *bipartite theorem*.

Theorem 2 (*[14, Theorem 3.4.2]*). *Let M be an infinite CFPO all of whose chains are finite. If M is k-CS-transitive for some $k \ge 2$ and C is a maximal chain in M, then $|C| = 2$.*

Note that if M has finite chains it does not necessarily follow that the completion M^D has finite chains. It follows from the above result that finite chain *CFPO*s can be thought of both as partial orders and as bipartite graphs.

A graph is *vertex transitive* if its automorphism group acts transitively on the set of vertices, and is *edge transitive* if its automorphism group acts transitively on the set of edges of the graph. An *s-arc* in a graph is a sequence $v_1, \dots, v_s$ of vertices such that v_i is adjacent to v_{i+1} for all $1 \le i \le s-1$, and $v_j \ne v_{j+2}$ for $1 \le j \le s-2$. A graph is *s-arc-transitive* if its automorphism group acts transitively on s-arcs. Clearly if a graph is 1-arc-transitive then it is edge transitive, but the converse is not true in general. For more background on these notions we refer the reader to [8, Chapters 3 and 4]. Let M be a poset with maximal chains of length 2, and let $\Gamma(M)$ be the corresponding bipartite graph. Clearly M is 2-CS-transitive if and only if $\Gamma(M)$ is edge transitive. If in addition, there is an anti-isomorphism of $(M, \le)$ interchanging the maximal and minimal points of M then $\Gamma(M)$ is arc-transitive. There is a similar relationship between 2-arc-transitivity in $\Gamma(M)$ and 3-CS-homogeneity in M.

One question arising from the work on *CFPO*s described above is to what extent this approach may be used in the investigation of countable k-arc-transitive graphs for $k \ge 1$. Given a countable k-arc-transitive bipartite graph we can, by defining one part of the bipartition to be above the other and so viewing it as a partial order, construct its D–M completion. Since the graph is 1-arc-transitive it follows that in the completion all the maximal intervals are isomorphic to some fixed interval I. Here by an *interval* we mean a poset I with elements $x, y \in I$ such that $x \le I \le y$. Thus, associated with any k-arc-transitive bipartite graph is such an interval, and the completion is constructed by gluing these together in a certain way. For example, if Γ is one of the *CFPO* bipartite graphs then this interval is a linear order. Conversely in [14, Chapter 4] it is shown exactly how these chains may be glued together to obtain *CFPO*s. Of course, many non-*CFPO*s will also give rise to linear orders as their intervals. For example, if Γ (thought of as a two-level poset) is already complete, the associated interval will just be a two-element chain. This happens for instance if the bipartite graph arises as the incidence graph of a generalized quadrangle. For lots of examples of this kind (i.e. continuum many) we refer the reader to [10]. From this is seems that there is not much hope of a classification of bipartite graphs whose completions have chain intervals. This demonstrates that it is not the fact that the intervals in the completion are chains that makes the class of *CFPO*s accessible; it is the simple way in which the intervals are glued together that is important. Here we investigate what happens when we allow the maximal intervals I in M^D to be something other than a chain, and our aim here is to give constructions of partial orders of this kind.

Given an interval P (a poset with a maximal and minimal element), and two functions ρ_u and ρ_d (each with domain P and range $\mathbb{N} \cup \{\aleph_0\}$) which we call upward and downward ramification functions, we want to construct a countable connected bipartite graph that is, at least, edge transitive and when viewed as a partial order Q, has intervals