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Graham's pebbling conjecture on products of many cycles

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Abstract

A pebbling move on a connected graph G consists of removing two pebbles from some vertex and adding one pebble to an adjacent vertex. We define $f_t(G)$ as the smallest number such that whenever $f_t(G)$ pebbles are on G, we can move t pebbles to any specified, but arbitrary vertex. Graham conjectured that $f_1(G \times H) \leq f_1(G)f_1(H)$ for any connected G and H. We define the α-pebbling number $\alpha(G)$ and prove that $\alpha(C_{p_j} \times \cdots \times C_{p_2} \times C_{p_1} \times G) \leq \alpha(C_{p_j}) \cdots \alpha(C_{p_2})\alpha(C_{p_1})\alpha(G)$ when none of the cycles is C_5, and G satisfies one more criterion. We also apply this result with $G = C_5 \times C_5$ by showing that $C_5 \times C_5$ satisfies Chung's two-pebbling property, and establishing bounds for $f_t(C_5 \times C_5)$.

Keywords: Pebbling; Graham's conjecture; Cartesian product; Cycle

1. Background

We show that the pebbling number of the Cartesian product of arbitrarily many cycles with a connected graph G satisfies Graham's conjecture as long as none of the cycles are C_5 and G satisfies Wang's *odd two-pebbling property* [10] and one more numerical criterion; that is, in this case, the pebbling number of the product is at most the product of the pebbling numbers of the individual cycles. We do this by defining the *alpha-pebbling number* of a graph, which is similar to notions used by Moews [7], and is based on the odd two-pebbling property. We also show that $C_5 \times C_5$ satisfies the necessary conditions. In this paper, we assume that all graphs are connected.

Chung [1] defined a *distribution* on a graph as a placement of pebbles on the vertices of the graph. A *pebbling move* then consists of removing two pebbles from one vertex, and adding one pebble to an adjacent vertex. Then the *pebbling number of a vertex* v *in* G is the smallest number $f(G, v)$ such that from every placement of $f(G, v)$ pebbles, it is possible to move a pebble to v by a sequence of pebbling moves. She also defined the t*-pebbling number of* v *in* G as the smallest number $f_t(G, v)$ such that from every placement of $f_t(G, v)$ pebbles, it is possible to move t pebbles to v. Then the *pebbling number of* G and the t*-pebbling number of* G are the smallest numbers, $f(G)$ and $f_t(G)$, such that from any placement of $f(G)$ pebbles or $f_t(G)$ pebbles, respectively, it is possible to move one or t pebbles, respectively, to any specified target vertex by a sequence of pebbling moves. Thus, $f(G)$ and $f_t(G)$ are the maximum values of $f(G, v)$ and $f_t(G, v)$ over all vertices v.

Chung also defined the *two-pebbling property* of a graph, and Wang [10] extended her definition to the *odd two-pebbling property* as follows.

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Definition 1. Given a distribution of pebbles on G, let p be the number of pebbles in that distribution, let q be the number of occupied vertices (vertices with at least one pebble), and let r be the number of vertices with an odd number of pebbles. We say that G satisfies the *two-pebbling property* (respectively, the *odd two-pebbling property*) if it is possible to move two pebbles to any specified target vertex whenever p and q satisfy the inequality $p + q > 2f(G)$ (respectively, whenever p and r satisfy $p + r > 2f(G)$). Clearly, any graph with the two-pebbling property also satisfies the odd two-pebbling property. It is not known whether any graph satisfies the odd two-pebbling property, but not the two-pebbling property.

Definition 2 is motivated by Moews' notion [7] of α-pebbling.

Definition 2. The *alpha-pebbling number of* v *in* G is the smallest number $\alpha(G, v)$ such that

(1) $f(G, v) \le \alpha(G, v)$, and
(2) From every placement of p pebbles on G in which r vertices are occupied by an odd number of pebbles, if $p + r > 2\alpha(G, v)$, then we can move two pebbles to v.

The *alpha-pebbling number of* G, $\alpha(G)$, is the maximum of $\alpha(G, v)$ over all vertices v. Thus, $f(G) \le \alpha(G)$, and the equality holds if and only if G satisfies the odd two-pebbling property. When we discuss graphs whose pebbling numbers are the same for every vertex (in particular, complete graphs, cycles, and products of these graphs), we generally do not specify the target vertex.

Chung attributed Conjecture 3 to Graham, and proved Theorem 4 ([1], Theorem 5), but the proof can be adapted to show Theorem 5, so we also attribute this result to Chung.

Conjecture 3 (*Graham*). *For any graphs G and H, if $G \times H$ represents the Cartesian product of G and H, then*

$$f(G \times H) \le f(G)f(H).$$

Theorem 4 (*Chung*). *Suppose G is a graph which satisfies the two-pebbling property, and let K_t denote the complete graph on t vertices. Then*

(1) $f(K_t \times G) \le tf(G)$,
(2) *If $f(K_t \times G) = tf(G)$, then $K_t \times G$ satisfies the two-pebbling property.*

Theorem 5 (*Chung*). *For any graph G, $\alpha(K_t \times G) \le \alpha(K_t)\alpha(G) = t\alpha(G)$.*

Chung used Theorem 4 to prove Graham's conjecture for products of K_t's, including hypercubes (products of K_2's). She also proved that trees satisfy the two-pebbling property (and therefore, the odd two-pebbling property). Moews [7] proved Theorem 6, though with slightly different notation.

Theorem 6 (*Moews*). *For any tree T, $\alpha(T \times G, (x, y)) \le \alpha(T, x)\alpha(G, y)$. In particular, if P_m is the path with m vertices, then $\alpha(P_m \times G, (x, y)) \le \alpha(P_m)\alpha(G, y) = 2^{m-1}\alpha(G, y)$.*

We call Conjecture 7 the alpha conjecture. It is a natural analog of Graham's conjecture.

Conjecture 7 (*The Alpha Conjecture*). *For any graphs G and H, and any vertex (x, y) in $G \times H$, we have $\alpha(G \times H) \le \alpha(G)\alpha(H)$, and $\alpha(G \times H, (x, y)) \le \alpha(G, x)\alpha(H, y)$.*

Conjecture 7 would imply Graham's conjecture for all graphs with the odd two-pebbling property.

There are several advantages to the notation of $\alpha(G)$. Much of the literature on pebbling [1,2,4,9,10] proves theorems of the form: *"If G is a certain type of graph, and H satisfies either the two-pebbling property or the odd two-pebbling property, then $f(G \times H) \le f(G)f(H)$"*. The proofs of the theorems which require H to satisfy the two-pebbling property apply with obvious modification even when H only satisfies the odd two-pebbling property. If we rewrite the conclusion to read "$f(G \times H) \le f(G)\alpha(H)$" (or if G satisfies the odd two-pebbling property we could write "$f(G \times H) \le \alpha(G)\alpha(H)$"), our result applies to all graphs H; if H does not satisfy the odd two-pebbling property, $G \times H$ may not obey Graham's conjecture, but the result is interesting nonetheless. Also, if we can prove