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On arc-traceable local tournaments

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Abstract

A digraph D is *arc-traceable* if for every arc xy of D, the arc xy belongs to a directed Hamiltonian path of D. A *local tournament* is an oriented graph such that the negative neighborhood as well as the positive neighborhood of every vertex induces a tournament. It is well known that every tournament contains a directed Hamiltonian path and, in 1990, Bang-Jensen showed the same for connected local tournaments. In 2006, Busch, Jacobson and Reid studied the structure of tournaments that are not arc-traceable and consequently gave various sufficient conditions for tournaments to be arc-traceable. Inspired by the article of Busch, Jacobson and Reid, we develop in this paper the structure necessary for a local tournament to be not arc-traceable. Using this structure, we give sufficient conditions for a local tournament to be arc-traceable and we present examples showing that these conditions are best possible.

Keywords: Arc-traceable; Hamiltonian path; Local tournaments

1. Terminology and introduction

For a digraph D, we denote by $V(D)$ and $E(D)$ the *vertex set* and *arc set* of D, respectively. The number $|V(D)|$ is the *order* of the digraph D. The subdigraph induced by a subset A of $V(D)$ is denoted by $D[A]$.

If $xy \in E(D)$, then y is a *positive neighbor* or *out-neighbor* of x and x is a *negative neighbor* or *in-neighbor* of y, and we also say that x *dominates* y and that y *is dominated by* x, denoted by $x \to y$. More generally, if A and B are two disjoint subdigraphs of a digraph D such that every vertex of A dominates every vertex of B, then we say that A *dominates* B and that B *is dominated by* A, denoted by $A \to B$. Furthermore, $A \rightsquigarrow B$ denotes the fact that there is no arc leading from B to A and at least one arc is leading from A to B. In this case also we say that A *weakly dominates* B. The *outset* $N^+(x)$ of a vertex x is the set of positive neighbors of x. More generally, for arbitrary subdigraphs A and B of D, the outset $N^+(A, B)$ is the set of vertices in B to which there is an arc from a vertex in A. The *insets* $N^-(x)$ and $N^-(A, B)$ are defined analogously. The numbers $d^+(x) = \left|N^+(x)\right|$ and $d^-(x) = \left|N^-(x)\right|$ are called *outdegree* and *indegree* of x, respectively. The *minimum outdegree* $\delta^+(D)$ and the *minimum indegree* $\delta^-(D)$ of D are given by $\min\left\{d^+(x)|x \in V(D)\right\}$ and $\min\left\{d^-(x)|x \in V(D)\right\}$, respectively. Furthermore, let $\delta(D)$ denote the minimum of $\delta^+(D)$ and $\delta^-(D)$.

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Throughout this paper, directed cycles and paths are simply called *cycles* and *paths*. The length of a cycle C or a path P is the number of arcs included in C or P. Let $C = x_1x_2 \dots x_kx_1$ be a cycle of length k. Then $C[x_i, x_j]$, where $1 \le i, j \le k$, denotes the subpath $x_ix_{i+1} \dots x_j$ of C with *initial vertex* x_i and *terminal vertex* x_j. Furthermore, if x is a vertex of C, then x_C^+ denotes the successor of x on C. The *predecessor* of a vertex x is defined analogously. If no confusion arises, x^+ and x^- will be used to denote x_C^+ and x_C^-. The notations for paths are defined analogously.

A digraph D is *arc-traceable* if every arc xy of D belongs to a path of order $|V(D)|$, i.e., a Hamiltonian path.

All digraphs mentioned here are finite without loops, multiple arcs and cycles of length two.

We speak of a *connected digraph* if the underlying graph is connected. A digraph D is said to be *strongly connected* or just *strong*, if for every pair x, y of vertices of D, there is a path from x to y. A *strong component* of D is a maximal induced strong subdigraph of D. A digraph D is *k-connected* if for any set S of at most $k - 1$ vertices the subdigraph $D - S$ is strong. If D is a strong digraph and S is a subset of $V(D)$ such that $D - S$ is not strong, we say that S is a *separating set*. We speak of a separating vertex s if $S = \{s\}$ is a separating set of size one. A separating set S is called *minimal separating set* (*minimum separating set*) if there exists no separating set U such that $U \subseteq S$ and $U \ne S$ ($|U| < |S|$).

An *n-tournament* is an orientation of a complete undirected graph with order n. A *local tournament* is a digraph where the inset as well as the outset of every vertex induces a tournament and an *in-tournament* is a digraph where the inset of every vertex induces a tournament.

A tournament T is called *regular* if $\delta(T) = \Delta(T)$ and *almost-regular* if $\Delta(T) - \delta(T) \le 1$.

Let D be a digraph with $V(D) = \{v_1, v_2, \dots, v_r\}$ and let $H_1, H_2, \dots, H_r$ be a collection of digraphs. Then $D[H_1, H_2, \dots, H_r]$ is the new digraph obtained from D by replacing each vertex v_i of D with H_i and adding the arcs from every vertex of H_i to every vertex of H_j if v_iv_j is an arc of D for all i and j satisfying $1 \le i \ne j \le r$. The following class of digraphs plays an important role in the study of local tournaments. A digraph on n vertices is called a *round digraph* if we can label its vertices $v_1, v_2, \dots, v_n$ such that $N^+(v_i) = \{v_{i+1}, v_{i+2}, \dots, v_{i+d^+(v_i)}\}$ and $N^-(v_i) = \{v_{i-1}, v_{i-2}, \dots, v_{i-d^-(v_i)}\}$ for every i, where the subscripts are taken modulo n. We refer to $v_1, v_2, \dots, v_n$ as a *round labeling* of D. A local tournament D is *round-decomposable* if there exists a round local tournament R on $r \ge 2$ vertices and strong local subtournaments $H_1, H_2, \dots, H_r$ of D such that $D = R[H_1, H_2, \dots, H_r]$. We call $R[H_1, H_2, \dots, H_r]$ a *round decomposition* of D.

Throughout this paper all subscripts are taken modulo the corresponding number.

In 1990, Bang-Jensen [2] defined local tournaments to be the family of oriented graphs where the inset as well as the outset of every vertex induces a tournament. In transferring the general adjacency only to vertices that have a common positive or a common negative neighbor, local tournaments form an interesting generalization of tournaments. Since then a lot of research has been done concerning local tournaments, or the more general class of *locally semicomplete digraphs*, where there might be cycles of length two. In particular, the Ph.D. Theses of Guo [9] and Huang [11] handled this subject in detail. For more information concerning different generalizations of tournaments, the reader may be referred to the survey article of Bang-Jensen and Gutin [3].

In claiming adjacency only for vertices that have a common positive neighbor, Bang-Jensen, Huang and Prisner [4] introduced a further generalization of local tournaments, the class of in-tournaments. Some problems concerning in-tournaments have been studied by Bang-Jensen, Huang and Prisner in their initial article [4].

In this paper, we develop the structure necessary for a local tournament D to contain an arc that does not belong to a Hamiltonian path. Using this structure, we give sufficient conditions for a local tournament to be arc-traceable. In addition we give examples that show that these conditions are best possible.

The first result concerning Hamiltonian paths in digraphs is due to Rédei [15] who showed that every tournament contains a Hamiltonian path.

Theorem 1.1 (*Rédei [15] 1934*). *Every tournament contains a Hamiltonian path.*

In 1990, Bang-Jensen [2] showed the same for connected local tournaments. In 1960, a first sufficient condition was given for a digraph to contain a Hamiltonian cycle (and thus, in particular a Hamiltonian path).

Theorem 1.2 (*Ghouila-Houri [8] 1960*). *If D is a digraph such that $\delta(D) \ge |V(D)|/2$, then D contains a Hamiltonian cycle.*

Moon [14] proved that every vertex in a strongly connected tournament belongs to cycles of arbitrary lengths.