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Note

Incidence coloring of the squares of some graphs

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Abstract

The incidence chromatic number of G, denoted by $\chi_i(G)$, is the least number of colors such that G has an incidence coloring. In this paper, we determine the incidence chromatic number of the powers of paths, trees, which are $\min\{n, 2k+1\}$, and $\Delta(T^2)+1$, respectively. For the square of a Halin graph, we give an upper bound of its incidence chromatic number.

Keywords: Incidence coloring; Square graph; Tree; Halin graph

1. Introduction

Graphs considered here are finite, undirected and simple. Let G be a graph. We denote by $V(G)$, $E(G)$ and $\Delta(G)$ its vertex set, edge set and maximum degree of G respectively. Let $N_G(v)$ be the set of neighbors of v in G and $d_G(v) = |N_G(v)|$ be its degree. Let S be a subset of $E(G)$ (or $V(G)$). The induced subgraph induced by S, $G[S]$, is the graph with edge set S (or vertex set S) and vertex set $\{x$: there is some $y \in V(G)$ such that $xy \in S.\}$ (or $\{xy : x, y \in S, \text{ and } xy \in E(G)\}$). Let S be a subset of $V(G)$, and $y \in V(G)$. The degree of y with respect to S is defined to be $d_G(y, S) = |N_G(y) \cap S|$. For vertices u, v in G, we let $\text{dist}_G(u, v)$ denote the distance between u and v, which is the length of the shortest path joining them. The diameter of G, denoted by $D(G)$, is the maximum value among $d_G(u, v)$ for any two vertices of G. The *square* of a graph G (denoted by G^2) is defined such that $V(G^2) = V(G)$, and two vertices u and v are adjacent in G^2 if and only if $d_G(u, v) \leq 2$. If 2 is replaced by k, we call the obtained graph the *kth power* of G.

An incidence in G is a pair (v, e) with $v \in V(G)$, $e \in E(G)$ such that v and e are incident. The set of all incidences of G is denoted by $I(G)$, that is $I(G) = \{(v, e) : v \in V, e \in E, v \text{ is incident with } e\}$. For a vertex v, we use $I(v)$ to denote the set of incidences of the form (v, vw) and use $A(v)$ to denote the set of incidences of the form (w, wv) respectively. Obviously, for each edge xy of G, there are two incidences with respect to xy, which are (x, xy) and (y, yx). For an incidence (x, xy), the edge xy is the edge with respect to the incidence (x, xy). Let S be a subset of $I(G)$, and $E(S)$ be the set of edges with respect to the elements of S. The subgraph induced by S is the subgraph induced by $E(S)$. Let F be a subset of $E(G)$. We use $I(F)$ to denote the subset of $I(G)$ with respect to the elements of F. For an incidence, we will use $(v, \overrightarrow{w})$ instead of (v, vw) for simplicity. The two incidences (v, e) and (w, f) are adjacent if one of the following hold: (1) $v = w$, (2) $e = f$; (3) $vw = e$ or f.

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An *incidence coloring* of a graph G is a mapping λ of $I(G)$ to a set C of colors such that adjacent incidences are assigned distinct colors. Such a coloring, sometimes, we call a proper incidence coloring. A *partial incidence coloring* of a graph is an incidence coloring which colors not all of its incidences. The *incidence chromatic number* of G, denoted by $\chi_i(G)$, is the least number of colors such that G has an incidence coloring. Let S be the set of incidences, λ be an incidence coloring of G. We use $\lambda(S)$ to denote the set of colors which are assigned to the elements of S. If all elements of S are assigned to the same color a, we use $\lambda(S) = a$ instead of $\lambda(S) = \{a\}$ to denote the color set. Let $P = x_1x_2\cdots x_n$ be a path of order n. If we use three colors 1, 2, 3 to color the incidence of P along the edges as follows: $\lambda(x_1, \overrightarrow{x_2}) = 1$, $\lambda(x_2, \overrightarrow{x_1}) = 2$, $\lambda(x_2, \overrightarrow{x_3}) = 3$, $\lambda(x_3, \overrightarrow{x_2}) = 1$, $\lambda(x_3, \overrightarrow{x_4}) = 2$, $\lambda(x_4, \overrightarrow{x_3}) = 3, \ldots$. We call this way of coloring a color pattern $(1, 2, 3, 1, 2, 3, \ldots)$ of P starting from x_1 and ending x_n. For a cycle of order n, do the same as that of P, and let $\lambda(x_n, \overrightarrow{x_1}) = \lambda(x_n, \overrightarrow{x_{n-1}}) + 1 \pmod 3$, and $\lambda(x_1, \overrightarrow{x_n}) = \lambda(x_n, \overrightarrow{x_1}) + 1 \pmod 3$. Note that λ may not be a proper incidence coloring unless the length of the cycle is a multiple of three.

The concept of incidence coloring was introduced by Bruadli and Massey [4]. It is easy to see that for any graph G with at least one edge, $\chi_i(G) \geq \Delta(G) + 1$. Bruadli and Massey in 1993 in [4] posed the incidence conjecture, which says that for any graph G with at least one edge, $\chi_i(G) \leq \Delta(G) + 2$. In 1997, Guiduli provided some counterexamples to this conjecture, and observed that the incidence coloring is a special case of directed star arboricity, introduced by Algor and Alon [1]. Bruadli and Massey showed that $\chi_i(G) \leq 2\Delta(G)$ for every graph. Guiduli in [8] proved that there exist graphs G with $\chi_i(G) \geq \Delta(G) + \Omega(\log \Delta(G))$, and proved the upper bound as follows $\chi_i(G) \leq \Delta(G) + O(\log \Delta(G))$.

Brundli and Massey determined the incidence chromatic number of trees, complete graphs and bipartite complete graphs. Chen et al. in [5,6], Huang in [10] and Liu and Li in [11] determined the incidence chromatic number of paths, cycles, fans, wheels, wheels with some more edges and complete tripartite graphs, Halin graphs, outerplanar graphs, Hamiltonian cubic graphs, the square of cycles, complete k-partite graphs etc. Dolama et al in [7] determined the incidence coloring number of K_4-minor free graphs and give an upper bound for k-degenerated graphs and planar graphs. M. Maydanskiy showed in [13] that the incidence chromatic number of a subcubic graph is at most five.

In Section 2, we show that the incidence chromatic number of P_n^k is $\min\{n, 2k+1\}$. In Section 3, we determine that the incidence chromatic number of T^2 is $\chi_i(T^2) = \Delta(T^2) + 1$. Section 4 concerns the incidence chromatic number of the square of a Halin graph, an upper bound $\Delta(T^2) + \Delta(T) + 8$ is given.

2. Powers of paths

Let $P_n = x_1x_2\ldots x_n$ be a path with n vertices. Obviously, its incidence chromatic number is 3. We can color the incidences with the color pattern $1, 2, 3, \ldots, 1, 2, 3$.

Lemma 2.1 ([4]). *Let K_n be the complete graph of order n. Then $\chi_i(K_n) = n$.*

Theorem 2.2. *Let n, k be integers. Then $\chi_i(P_n^k) = n$ if $n \leq 2k + 1$, otherwise $\chi_i(P_n^k) = 2k + 1$.*

Proof. If $n \leq k + 1$, P_n^k is isomorphic to the complete graph K_n. By Lemma 2.1, we have $\chi_i(P_n^k) = n$. If $k + 2 \leq n \leq 2k$, $\Delta(P_n^k) = n - 1$. Since P_n^k is a subgraph of K_n, $n \leq \chi_i(P_n^k) \leq \chi_i(K_n) = n$.

If $n > 2k + 1$, $\Delta(P_n^k) = 2k$. It suffices for us to give a $(2k + 1)$-incidence coloring of P_n^k. We define the $(2k + 1)$-incidence coloring λ of $I(P_n^k)$ as follows: $\lambda(A(x_i)) = i \bmod (2k + 1)$, where $i = 1, 2, \ldots, n$.

Now we check that λ is a proper incidence coloring. For integers i, j and $i < j$, and x_i is adjacent to x_j. Then $1 \leq j - i \leq k$, and $j \notin \lambda(A(x_i))$, and $i \notin \lambda(A(x_j))$. $\lambda(x_i, \overrightarrow{x_j}) = j \neq \lambda(x_j, \overrightarrow{x_i}) = i$. For integers i, j, r with $i < j < r$, and x_i is adjacent to x_j, x_j is adjacent to x_r, we need to show that $\lambda(x_i, \overrightarrow{x_j}) \neq \lambda(x_j, \overrightarrow{x_r})$. By definition, $\lambda(x_i, \overrightarrow{x_j}) = j \bmod (2k + 1)$, $\lambda(x_j, \overrightarrow{x_r}) = r \bmod (2k + 1)$. $j \neq r \bmod (2k + 1)$ since $1 \leq j - r \leq k$. $\lambda(x_j, \overrightarrow{x_i}) \neq \lambda(x_j, \overrightarrow{x_r})$. Since $1 \leq r - j \leq k$, $1 \leq j - i \leq k$, $2 \leq r - i = r - j + j - i \leq 2k$. and $r \neq i \bmod (2k + 1)$. Then it is a proper incidence coloring. □

3. Square of a tree

We call the complete bipartite graph $K_{1,n}$ a *star* with center x which is adjacent to all the other vertices. Let K_{1,r_1} and K_{1,r_2} be two stars with center x and y respectively. After joining an edge xy, we call the obtained graph a *double star* with center x and y.