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Note

A result on combinatorial curvature for embedded graphs on a surface

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Abstract

Let G be an infinite graph embedded in a surface such that each open face of the embedding is homeomorphic to an open disk and is bounded by finite number of edges. For each vertex x of G, we define the combinatorial curvature

$$K_G(x) = 1 - \frac{d(x)}{2} + \sum_{\sigma \in F(x)} \frac{1}{|\sigma|},$$

where $d(x)$ is the degree of x, $F(x)$ is the multiset of all open faces σ in the embedding such that the closure $\bar{\sigma}$ contains x, and $|\sigma|$ is the number of sides of edges bounding the face σ. In this paper, for a finite simple graph G embedded in a surface with $3 \leq d_G(x) < \infty$ and $K_G(x) > 0$ for all $x \in V(G)$, we have (i) if G is embedded in a projective plane and $|V(G)| = n \geq 290$, then G is isomorphic to P_n; (ii) if G is embedded in a sphere and $|V(G)| = n \geq 580$, then G is isomorphic to either A_n or B_n.

Keywords: Combinatorial curvature; Gauss–Bonnet formula; Euler relation; Infinite graph; Embedding; Face cycle; Finiteness theorem

1. Introduction

The notion of combinatorial curvature was introduced by Gromov [6] to study hyperbolic groups. Later it was modified by Ishida [8]. Then, the notion of discrete curvature was considered by some others, for example, see [1–4]. Moreover, using the curvature of point, Imiya et al. [7] got the application for image processing. We will give some notations used in this paper.

Let G be a graph embedded in a compact surface S without boundary. The graph G can be infinite, and may have loops and multiple edges; however, each vertex is required a finite degree. We view the vertex set $V(G)$ as a subset of S, each edge of G as an open arc of S, and consider G as the union $V(G) \bigcup E(G)$ so that G is a subset of S. If $V(G)$ is infinite, the accumulation set

$$V'(G) := \overline{G} - G$$

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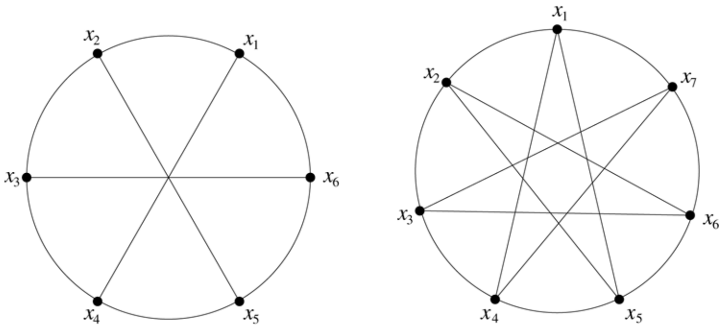

Fig. 1. The projective wheel graphs P_6 and P_7.

may not be empty, where $\overline{G}$ is the closure of the subset G in S. To avoid pathological cases, we assumed that the embedding satisfies the following properties:

(C1) The accumulation set $V'(G)$ is finite;
(C2) The complement $S - \overline{G}$ is a disjoint union of connected open sets, each such open set U is homeomorphic to an open disk and its boundary ∂U in S ($\partial U = \overline{U} - U$) is a finite subgraph of G.

Then the punctured surface $S - V'(G)$ is decomposed into a collection of (possibly infinitely many) vertices, open edges, and open regions. We call each open region an *open face* (or just a *face*) of G, and call each accumulation point in $V'(G)$ an *end* of G.

Note that the closure of a face may not be homeomorphic to a closed disk. This means that the boundary of a face may not be a cycle of G. Since each edge of G in the surface has two sides, we say that one side of an edge bounds a face σ provided that σ is exactly on that side of the edge. The *length* of a face σ is the number of sides of σ, and is denoted by $|\sigma|$.

For each vertex x of G, we denote by $d_G(x)$ or just $d(x)$ the *degree* of x (the number of edges incident with x), and by $F(x)$ the multiset of faces σ such that x is contained in the closure $\bar{\sigma}$; the multiplicity of a face σ is the number of times that x is visited when one travels along the sides of σ in an orientation.

Definition 1.1. Let G be a graph (finite or infinite) embedded in a compact surface S without boundary, satisfying the conditions (C1) and (C2). The *combinatorial curvature* of G is the function $K_G : V(G) \longrightarrow \mathbf{R}$ given by

$$K_G(x) = 1 - \frac{d(x)}{2} + \sum_{\sigma \in F(x)} \frac{1}{|\sigma|}, \quad x \in V(G). \tag{1.1}$$

The number $K_G(x)$ is called the *curvature* of G at the vertex x.

The most interesting and important problem about the combinatorial curvature is perhaps to classify embedded graphs whose curvatures satisfy certain properties. We are interested in classifying the embedded graphs with positive curvature at every vertex. To state our result on such classification of finite graphs with positive curvature everywhere, we introduce a type of projective wheel graphs P_n with $n \geq 3$ vertices, and two types of *cylinder graphs* A_n and B_n for integers $n \geq 3$ vertices. The vertex set of P_n is

$$V(P_n) = \{x_1, x_2, \ldots, x_n\}$$

and the edge set of P_n is given as follows: For odd $n = 2s + 1$,

$$E(P_{2s+1}) = \{x_i x_{i+1}, x_i x_{s+i},\ x_i x_{s+i+1} : 1 \leq i \leq s + 1\},$$

where $x_i = x_{n+i}$, and for even $n = 2s$,

$$E(P_{2s}) = \{x_i x_{i+1}, x_i x_{s+i} : 1 \leq i \leq s\}.$$

The examples for P_n are demonstrated in Fig. 1.