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Ramsey regions

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Abstract

Let $(T_1, T_2, \ldots, T_c)$ be a fixed c-tuple of sets of graphs (i.e. each T_i is a set of graphs). Let $R(c, n, (T_1, T_2, \ldots, T_c))$ denote the set of all n-tuples, $(a_1, a_2, \ldots, a_n)$, such that every c-coloring of the edges of the complete multipartite graph, $K_{a_1,a_2,\ldots,a_n}$, forces a monochromatic subgraph of color i from the set T_i (for at least one i). If $\mathbb{N}$ denotes the set of non-negative integers, then $R(c, n, (T_1, T_2, \ldots, T_c)) \subseteq \mathbb{N}^n$. We call such a subset of $\mathbb{N}^n$ a "Ramsey region". An application of Ramsey's Theorem shows that $R(c, n, (T_1, T_2, \ldots, T_c))$ is non-empty for $n \geqslant 0$. For a given c-tuple, $(T_1, T_2, \ldots, T_c)$, known results in Ramsey theory help identify values of n for which the associated Ramsey regions are non-empty and help establish specific points that are in such Ramsey regions. In this paper, we develop the basic theory and some of the underlying algebraic structure governing these regions.

Keywords: Ramsey theory; Designs; Multipartite graphs

1. Introduction

Ramsey theory dates back 75 years to the following theorem:

Theorem 1 (*Ramsey*). *Let r, k, l be given positive integers. There exists a positive integer n with the following property. If the k-subsets of an n element set are colored with r colors then there exists an l element set all of whose k-subsets are of the same color.*

For a given r, k, l it is an interesting (and hard) problem to find the smallest value of n guaranteed to exist by Ramsey's Theorem. The theorem has many corollaries guaranteeing the existence of substructures under various conditions. If k is a set equal to 2 then Ramsey's Theorem is a theorem in graph theory. It states that for n sufficiently large, any r coloring of the edges of K_n contains a monochromatic subgraph isomorphic to K_l. Since any graph, G, embeds in some complete graph, the theorem also implies that for n sufficiently large, any r coloring of the edges of K_n contains a monochromatic subgraph isomorphic to G.

In this paper, $K_{a_1,a_2,\ldots,a_t}$ will denote the complete t-partite graph on sets of vertices of size $a_1, a_2, \ldots, a_t$ and K_n will denote the complete graph on n vertices (thus $K_n = K_{(1,1,\ldots,1)}$). Let G be an arbitrary graph. If the vertices of G can be colored with t colors such that adjacent vertices have different colors then G can be embedded into a complete

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t-partite graph. The smallest t such that G can be embedded into a complete t-partite graph is called the chromatic number of G and is denoted by $\chi(G)$.

Let $(T_1, T_2, \ldots, T_c)$ be a c-tuple of sets of graphs (i.e. each T_i is a *set* of graphs). Let $R(c, n, (T_1, T_2, \ldots, T_c))$ denote the set of all n-tuples such that every c-coloring of the edges of $K_{a_1,a_2,\ldots,a_n}$ forces a monochromatic subgraph of color i from the set T_i (for some i). Ramsey's Theorem guarantees that $R(c, n, (T_1, T_2, \ldots, T_c))$ is non-empty for $n \gg 0$. More precisely, Ramsey's Theorem guarantees that the n-tuple consisting entirely of 1's is an element of $R(c, n, (T_1, T_2, \ldots, T_c))$ provided that $n \gg 0$. The set of all n-tuples in $R(c, n, (T_1, T_2, \ldots, T_c))$ is called a *Ramsey region*. The goal of this paper is to develop the basic theory of Ramsey regions. Let M denote the minimum value of the chromatic numbers of the graphs in the various T_i's. M provides a lower bound on n such that $R(c, n, (T_1, T_2, \ldots, T_c))$ is non-empty. Known results in Ramsey theory can be used to give upper bounds on n such that the n-tuple consisting entirely of 1's lies in $R(c, n, (T_1, T_2, \ldots, T_c))$. In general, these bounds are far from being sharp. However, we can use many of the bounds found in [11] as a starting point for bounding Ramsey regions. The result and ideas found in [1–3] and in [5–10] provided much of the original impetus to formulate a setting within which a broad range of Ramsey theoretical results could sit and influence each other. It is important to note that any given Ramsey region can be described completely by a *finite* list of n-tuples(even though a non-empty Ramsey region will have an infinite number of points). Part of the motivation for introducing Ramsey regions arose when addressing the complexity issues in computing new Ramsey numbers. With Ramsey regions, a complex problem can be broken down into smaller pieces. Furthermore, information learned about one Ramsey region can help in the understanding of other Ramsey regions. As a result, it is hoped that an accumulation of knowledge can build to allow for the solution of problems that would be difficult to attack in a single step.

2. Main definitions and theorems

For this entire section, $K_{a_1,a_2,\ldots,a_n}$ will denote a complete multipartite graph and $A = (T_1, T_2, \ldots, T_c)$ will be an ordered c-tuple of sets of graphs. All graphs are assumed to have no multiple edges and no loops. $R(c, n, A)$ will denote the set of all n-tuples such that every c-coloring of the edges of $K_{a_1,a_2,\ldots,a_n}$ forces a monochromatic subgraph of color i which is isomorphic to a graph from the set T_i (for at least one value of i). If $T = T_1 = T_2 = \cdots = T_c$ then we write $R(c, n, T)$ instead of $R(c, n, (T_1, T_2, \ldots, T_c))$.

Proposition 2. *Suppose $K_{a_1,a_2,\ldots,a_n} \subseteq K_{b_1,b_2,\ldots,b_m}$ then*

$$(a_1, a_2, \ldots, a_n) \in R(c, n, A) \Longrightarrow (b_1, b_2, \ldots, b_m) \in R(c, m, A).$$

Proof. Fix an injection $K_{a_1,a_2,\ldots,a_n} \hookrightarrow K_{b_1,b_2,\ldots,b_m}$. In coloring the edges of $K_{b_1,b_2,\ldots,b_m}$ with c colors, you induce a coloring of the edges of $K_{a_1,a_2,\ldots,a_n}$ with c colors. Thus, if there exists a monochromatic subgraph of $K_{b_1,b_2,\ldots,b_m}$ of color i which is isomorphic to a given graph G then the induced coloring of the edges of $K_{a_1,a_2,\ldots,a_n}$ will also contain a monochromatic subgraph of color i which is isomorphic to G. □

Corollary 3. *Let S_n denote the symmetric group on n elements and suppose $\pi \in S_n$, then*

$$(a_1, a_2, \ldots, a_n) \in R(c, n, A) \iff (a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(n)}) \in R(c, n, A).$$

Proof. Follows from Proposition 2 using $K_{a_1,a_2,\ldots,a_n} \simeq K_{a_{\pi(1)},a_{\pi(2)},\ldots,a_{\pi(n)}}$. □

Corollary 4. *If $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are n-tuples such that $b_i \geqslant a_i$ for $1 \leqslant i \leqslant n$ then*

$$(a_1, a_2, \ldots, a_n) \in R(c, n, A) \Longrightarrow (b_1, b_2, \ldots, b_n) \in R(c, n, A).$$

Proof. Follows from Proposition 2 using $K_{a_1,a_2,\ldots,a_n} \subseteq K_{b_1,b_2,\ldots,b_n}$. □

Corollary 5. *If $(a_1, a_2, \ldots, a_n)$ is an n-tuple and b satisfies $0 \leqslant b \leqslant a_1$ then*

$$(a_1, \ldots, a_n) \in R(c, n, A) \Longrightarrow (a_1 - b, a_2, \ldots, a_n, b) \in R(c, n+1, A).$$

Proof. Follows from Proposition 2 using $K_{a_1,a_2,\ldots,a_n} \subseteq K_{a_1-b,a_2,\ldots,a_n,b}$. □