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Paths partition with prescribed beginnings in digraphs: A Chvátal–Erdős condition approach

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Abstract

A digraph D verifies the Chvátal–Erdős conditions if $\alpha(D) \leqslant \kappa(D)$, where $\alpha(D)$ is the stability number of D and $\kappa(D)$ is its vertex-connectivity. Related to the Gallai–Milgram Theorem (see Gallai and Milgram [Verallgemeinerung eines Graphentheorischen Satzes von Redei, Acta Sci. Math. 21 (1960) 181–186]), we raise in this context the following conjecture. For every set of $\alpha = \alpha(D)$ vertices $\{x_1, \ldots, x_\alpha\}$, there exists a vertex-partition of D into directed paths $\{P_1, \ldots, P_\alpha\}$ such that P_i begins at x_i for all i. The case $\alpha(D) = 2$ of the conjecture is proved.

Keywords: Digraphs; Vertex-partition; Vertex-connectivity; Chvátal–Erdős conditions

1. Introduction

All topics of the paper deal with digraphs. Considered paths and circuits are directed ones. In our digraphs, circuits of length 2 are allowed, but not loops. Classical definitions on digraphs may be found in [1,2].

A digraph D is *strongly connected* (or *strong* for short) if for every vertices x and y of D there exists a path from x to y in D. For an integer k, D is *k-strongly connected* (or k-strong) if D has at least $k+1$ vertices and if for every k-set of vertices $\{x_1, \ldots, x_k\}$, $D \backslash \{x_1, \ldots, x_k\}$ is strong. The maximum k for which D is k-strong is denoted by $\kappa(D)$, the *vertex-connectivity* of D.

A subset X of vertices of D is a *stable set* (or *independent set*) if the vertices of X are pairwise not linked by an arc. We denote $\alpha(D)$ the *stability number* of D, which is the maximum cardinality of a stable set of D.

All partitions or coverings of digraphs mentioned in the paper are understood as vertex partitions or coverings.

The classical Gallai–Milgram Theorem (see [7]) states that every digraph admits a partition into α paths. In this paper, we are mainly concerned by finding conditions to prescribe the beginnings of paths in such a partition. This problem is motivated by extending a result holding for non-oriented graphs (Theorem 1) and, in a remote way, by covering of digraphs with circuits.

The following definitions are given for a digraph D with vertex set V and arc set E. For a path P of D, we denote by $b(P)$ and $e(P)$, respectively, its *beginning* and its *end*. The *internal vertices* of P are the vertices of $P \backslash \{b(P), e(P)\}$ (possibly empty). For two vertices x and y of D, an (x, y)-*path* is a path with beginning x and end y. By extension, an

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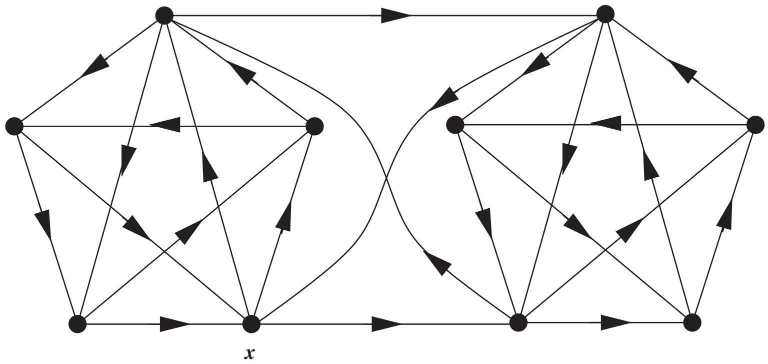

Fig. 1. Digraph with $\alpha \leqslant \kappa = 2$ and no Hamilton path beginning at x.

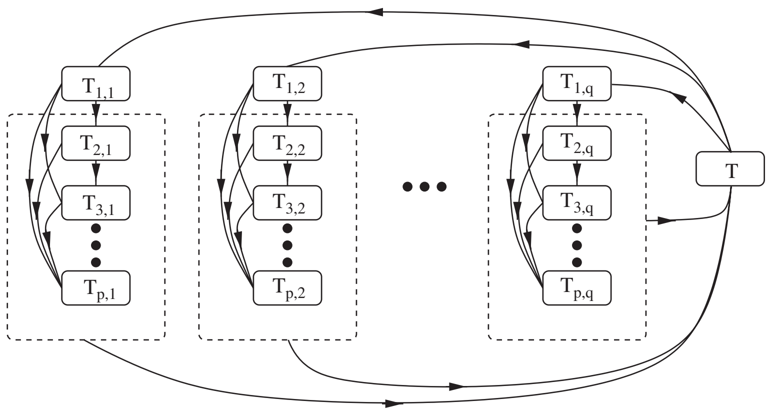

Fig. 2. Extremal family of digraph for Conjecture 1 (an arc from a 'box' to another one stands for all the arcs from the first one to the second one).

(X, Y)-*path* P is an (x, y)-path for some $x \in X$ and $y \in Y$ such that the set of internal vertices of P and $X \cup Y$ are disjoint.

For a path P and a vertex x of P, xP (resp. Px) denote the maximal sub-path of P which starts (resp. ends) at x. Moreover, if y is a vertex of xP, xPy denotes the maximal sub-path of xP which ends in y (i.e. the sub-path of P which starts at x and ends at y). We denote the concatenation of two paths by . ($P.Q$ is only used when there exists an arc from the end of P to the beginning of Q).

Finally, for an arc $xy \in E$ we also denote by xy the path of length 1 from x to y.

A digraph D verifies the *Chvátal–Erdős conditions* if we have $\alpha(D) \leqslant \kappa(D)$. These were named from the following sufficient condition for a (non-oriented) graph to have a Hamilton cycle, given by Chvátal and Erdős in 1972. For a non-oriented graph G, $\kappa(G)$ and $\alpha(G)$ match to the corresponding parameters for the symmetrized digraph of G, obtained by substituting each (non-oriented) edge of G with two opposite arcs.

Theorem 1 (*Chvátal–Erdős [6]*). *For a graph G, if $\alpha(G) \leqslant \kappa(G)$, then G has a Hamilton cycle.*

However, for digraphs the condition $\alpha \leqslant \kappa$ does not imply the existence of Hamilton circuit. Infinite families of examples for $\alpha = 2$ and 3 are given in [9], there are generalizations of the digraphs given in Figs. 1 and 3. However, according to the previous result for graphs, it could seem possible to ask for special partitions into paths in digraphs which satisfy Chvátal–Erdős conditions. Several results and conjectures are stated in a survey of Jackson and Ordaz (see [9]). We present a new conjecture in this area.